Symmetry of matrix-valued stochastic processes and noncolliding diffusion particle systems

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As an extension of the theory of Dyson's Brownian motion models for the standard Gaussian random-matrix ensembles, we report a systematic study of hermitian matrix-valued processes and their eigenvalue processes associated with the chiral and nonstandard random-matrix ensembles. In addition to the noncolliding Brownian motions, we introduce a one-parameter family of temporally homogeneous noncolliding systems of the Bessel processes and a two-parameter family of temporally inhomogeneous noncolliding systems of Yor's generalized meanders and show that all of the ten classes of eigenvalue statistics in the Altland-Zirnbauer classification are realized as particle distributions in the special cases of these diffusion particle systems. As a corollary of each equivalence in distribution of a temporally inhomogeneous eigenvalue process and a noncolliding diffusion process, a stochastic-calculus proof of a version of the Harish-Chandra (Itzykson-Zuber) formula of integral over unitary group is established.

I INTRODUCTION

It is interesting to consider today mathematical-physical sequences of the two classic papers [11] and [10] by Dyson of random matrix theory, which appeared sequentially in the same volume of the journal in 1962. In one of them [11], following the early work of Wigner, he gave a logical foundation for his classification scheme of random-matrix ensembles based on the group representation theory of Weyl and established the standard (Wigner-Dyson) random matrix theory for the three ensembles called the Gaussian unitary, orthogonal, and symplectic ensembles (GUE, GOE, and GSE). He introduced in the other paper [10] the hermitian matrix-valued Brownian motions, which are associated with these Gaussian random-matrix ensembles, and studied the stochastic processes of eigenvalues of the matrix-valued processes. Combining the standard perturbation theory of the quantum mechanics and a simple but essential consideration of the scaling of Brownian motions, he generally proved that the obtained eigenvalue processes are identified with the one-dimensional systems of Brownian particles with the repulsive two-body forces proportional to the inverse of distances between particles. These processes are now called Dyson's Brownian motion models $\mathbf{Y}(t) = (Y_1(t), Y_2(t), \cdots, Y_N(t))$ described by the stochastic differential equations

$$dY_i(t) = dB_i(t) + \frac{\beta}{2} \sum_{1 \le j \le N, j \ne i} \frac{1}{Y_i(t) - Y_j(t)} dt, \quad t \in [0, \infty), 1 \le i \le N, \tag{1}$$

with $\beta = 1, 2, 4$ for GOE, GUE and GSE, respectively, where $B_i(t), 1 \le i \le N$ are independent one-dimensional standard Brownian motions. Dyson's classification scheme has been extended. In addition to the standard three random-matrix ensembles, their *chiral versions* (chGUE, chGOE, and chGSE) were studied in the particle physics of QCD associated with consideration of the gauge groups and quantum numbers called flavors [54, 53, 27, 51]. After that extension, Altland and Zirnbauer introduced more four ensembles called the classes C, CI, D, and DIII for the solid-state physics of mesoscopic systems considering the particle-hole symmetry, which plays an important role in the Bogoliubov-de Gennes framework of the BCS mean-field theory of superconductivity[1, 2]. These totally ten Gaussian ensembles are systematically

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argued by Zirnbauer [56] based on Cartan's classification scheme of symmetric spaces [23] and Efetov's supersymmetry theory [13].

One consequence of a combination of the two papers by Dyson may be to give a systematic study of matrix-valued diffusion processes (*i.e.* diffusion processes in groups or algebraic spaces) and perform the classification of eigenvalue processes as generalization of Dyson's Brownian motion models. This line has been taken by Bru [6, 7], Grabiner [21], König and O'Connell [36] and others, and one of the purpose of the present paper is to clarify the relationship between statistics of (nonstandard) random matrix theory and stochastic processes of interacting diffusion particles in the type of Dyson's Brownian motion models studied in the probability theory. We will claim in Sec.II that the matrix-valued processes called the Wishart process by Bru [7] and the Laguerre process by König and O'Connell [36] are the stochastic versions of chGOE and chGUE, respectively, in the sense of Dyson [10], and derive in Sec.III the diffusion processes describing the eigenvalue statistics of the classes C and D of Altland and Zirnbauer, following Bru's matrix-version of the stochastic calculus based on the Ito rule for differentials.

Due to the strong repulsive forces in the processes of the types of Dyson's Brownian motion models, particle collisions are suppressed. Impossibility of collision may be generally proved by the same argument as Bru, who showed that the collision time between two eigenvalues of the Wishart process is infinite ($\tau = +\infty$ a.s.) [6]. For the $\beta = 2$ (GUE) case of Dyson's Brownian motion model (1), if $\mathbf{Y}(0) \in \mathbb{W}_N^{\mathrm{A}}$ then $\mathbf{Y}(t) \in \mathbb{W}_N^{\mathrm{A}}$ for all $t > 0$ with probability 1, where $\mathbb{W}_N^{\mathrm{A}}$ denotes the Weyl chamber of type A_{N-1}; $\mathbb{W}_N^{\mathrm{A}} = \{\mathbf{x} \in \mathbb{R}^N; x_1 < x_2 < \cdots < x_N\}$. Using the Karlin-McGregor formula [28, 29] the transition density of the absorbing Brownian motion in $\mathbb{W}_N^{\mathrm{A}}$ from the state $\mathbf{x}$ at time s to the state $\mathbf{y}$ at time $t(> s)$ is given by the determinant

$$f^{\mathrm{A}}(t-s, \mathbf{y}|\mathbf{x}) = \det_{1 \leq i,j \leq N} \left[G^{\mathrm{A}}(t-s, y_j|x_i) \right], \quad \mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\mathrm{A}}, \tag{2}$$

where each element is the Gaussian heat-kernel $G^{\mathrm{A}}(t, y|x) = e^{-(x-y)^2/2t}/\sqrt{2\pi t}$. Grabiner [21] pointed out that the transition probability density of the process (1) with $\beta = 2$ is given by

$$p^{\mathrm{A}}(s, \mathbf{x}; t, \mathbf{y}) = \frac{1}{h^{\mathrm{A}}(\mathbf{x})} f^{\mathrm{A}}(t-s, \mathbf{y}|\mathbf{x}) h^{\mathrm{A}}(\mathbf{y}),$$

where $h^{\mathrm{A}}(\mathbf{x}) = \prod_{1 \leq i < j \leq N}(x_j - x_i)$. Since $h^{\mathrm{A}}(\mathbf{x})$ is a strictly positive harmonic function in $\mathbb{W}_N^{\mathrm{A}}$, this is regarded as the h-transform in the sense of Doob [9], and it implies that the eigenvalue process of GUE is realized as the noncolliding Brownian motions (*i.e.* the h-transform of an absorbing Brownian motion in the Weyl chamber of type A_{N-1}). König and O'Connell also showed that the eigenvalue process of the Laguerre process, which corresponds to chGUE, is realized as the noncolliding system of the squared Bessel processes [36]. In Sec.IV, we show that the eigenvalue processes of random matrices in the symmetry classes C and D of Altland and Zirnbauer are realized as the noncolliding system of the *Brownian motions with an absorbing wall at the origin* [35] (*i.e.* the h-transform of an absorbing Brownian motion in the Weyl chamber of type C_N) and as the noncolliding system of the *reflecting Brownian motions* (*i.e.* the h-transform of an absorbing Brownian motion in the Weyl chamber of type D_N), respectively. These three kinds of systems are discussed as special cases of a family of noncolliding systems of diffusion particles with one parameter $\nu > -1$, in which each particle is following the $d = 2(\nu+1)$-dimensional Bessel process defined by the transition probability density [5, 48]

$$\begin{aligned} G^{(\nu)}(t, y|x) &= \frac{y^{\nu+1}}{x^\nu} \frac{1}{t} e^{-(x^2+y^2)/2t} I_\nu\left(\frac{xy}{t}\right) \quad \text{for} \quad x > 0, y \geq 0, \\ G^{(\nu)}(t, y|0) &= \frac{y^{2\nu+1}}{2^\nu \Gamma(\nu+1) t^{\nu+1}} e^{-y^2/2t} \quad \text{for} \quad y \geq 0, \end{aligned} \tag{3}$$

where Γ denotes the Gamma function and I_ν is the modified Bessel function; $I_\nu(z) = \sum_{n=0}^\infty (z/2)^{2n+\nu}/\{\Gamma(n+1)\Gamma(\nu+n+1)\}$.

How can we realize other six eigenvalue processes in Altland-Zirnbauer's ten classes of random-matrix ensembles as well by noncolliding systems of diffusion processes ? In our previous papers [31, 32] we considered the situation that the noncolliding condition is imposed not forever but for a finite time-interval $(0, T]$

to define the temporally inhomogeneous noncolliding Brownian motions $\mathbf{X}(t) = (X_1(t), X_2(t), \cdots, X_N(t))$. Of course, we can see that $\mathbf{X}(t) \to \mathbf{Y}(t)$ in distribution as $T \to \infty$. We observed for the finite time-interval $t \in [0,T]$ that, if we set $\mathbf{X}(0) = \mathbf{Y}(0) = \mathbf{0}$ with $\mathbf{0} = (0,0,\cdots,0) \in \mathbb{R}^N$, then

$$P(\mathbf{X}(\cdot) \in d\mathbf{w}) = \frac{C[\mathrm{A}]T^{\psi[\mathrm{A}]}}{C[\mathrm{A}']h^{\mathrm{A}}(\mathbf{w}(T))} P(\mathbf{Y}(\cdot) \in d\mathbf{w}), \tag{4}$$

where $C[\mathrm{A}] = (2\pi)^{N/2} \prod_{i=1}^N \Gamma(i)$, $C[\mathrm{A}'] = 2^{N/2} \prod_{i=1}^N \Gamma(i/2)$, and $\psi[\mathrm{A}] = N(N-1)/4$. This is regarded as a multivariate version of the Imhof relation in the probability theory [25], since it implies the absolute continuity in distribution of the temporally homogeneous process $\mathbf{Y}(t)$ and the inhomogeneous process $\mathbf{X}(t)$ in $[0,T]$, but from the viewpoint of random matrix theory the important consequence of this equality is the fact that the process $\mathbf{X}(t)$ exhibits a transition in distribution from the eigenvalue statistics of GUE to that of GOE and thus the GOE distribution is realized at the final time $t = T$. In Sec.V, we develop this argument by replacing the Brownian motions $X_i(t), 1 \leq i \leq N$ by the *generalized meanders* with two parameters $(\nu, \kappa), \nu > -1, \kappa \in [0, 2(\nu+1))$, introduced as the temporally inhomogeneous diffusions associated with the Bessel process by Yor [55], whose transition probability density is given by

$$G_T^{(\nu,\kappa)}(s,x;t,y) = \frac{1}{h_T^{(\nu,\kappa)}(s,x)} G^{(\nu)}(t-s,y|x) h_T^{(\nu,\kappa)}(t,y) \tag{5}$$

for $0 \leq s < t \leq T, x, y \geq 0$ with $h_T^{(\nu,\kappa)}(t,x) = \int_0^\infty dz\, G^{(\nu)}(T-t,z|x) z^{-\kappa}$. By choosing the two parameters (ν,κ) appropriately, this family of noncolliding systems of generalized meanders provides such diffusion processes that exhibit the transitions from chGUE to chGOE and from the class C to the class CI. We will also consider the processes, in which the noncolliding condition collapses at the final time $t = T$ in the ways that all particles collide simultaneously or only pairwise collisions occur. In the special cases in the latter situation, we have the processes showing the transitions from GUE to GSE, from chGUE to chGSE, and from the class D to the class DIII.

The present study of the temporally inhomogeneous noncolliding diffusion processes gives two kinds of byproducts. (i) Topology of path-configurations of our processes on the spatio-temporal plane $\mathbb{R} \times [0,T]$ is determined by the conditions at $t = 0$ and $t = T$. We will be able to discuss the topology of random directed polymer networks [8, 14] using the random matrix theory. Such correspondence between the topology of path-configurations and random-matrix ensembles is recently used by Sasamoto and Imamura to analyze one-dimensional polynuclear growth models [49]. (ii) A variety of versions of Harish-Chandra (Itzykson-Zuber) formulae of integrals over unitary groups [22, 26] are derived as corollaries of the equivalence in distribution of the eigenvalue processes of matrix-valued processes and noncolliding diffusion processes. Other remarks are given in Sec.VI.

II BRU'S THEOREM

A Hermitian matrix-valued stochastic processes

We denote the space of $N \times N$ hermitian matrices by $\mathcal{H}(N)$, the group of $N \times N$ unitary matrices by $\mathrm{U}(N)$, and the group of $N \times N$ real orthogonal matrices by $\mathrm{O}(N)$. We also use the notations $\mathcal{S}(N)$ and $\mathcal{A}(N)$ for the spaces of $N \times N$ real symmetric and real antisymmetric matrices, respectively. We consider complex-valued processes $\xi_{ij}(t) \in \mathbb{C}, 1 \leq i,j \leq N, t \in [0,\infty)$, with the condition $\xi_{ji}(t)^* = \xi_{ij}(t)$, and define the matrix-valued processes by $\Xi(t) = (\xi_{ij}(t))_{1 \leq i,j \leq N} \in \mathcal{H}(N)$. We denote by $U(t) = (u_{ij}(t))_{1 \leq i,j \leq N}$ the family of unitary matrices which diagonalize $\Xi(t)$ so that

$$U(t)^\dagger \Xi(t) U(t) = \Lambda(t) = \mathrm{diag}\{\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t)\},$$

where $\{\lambda_i(t)\}_{i=1}^N$ are eigenvalues of $\Xi(t)$ and we assume their increasing order

$$\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_N(t). \tag{6}$$

Define $\Gamma_{ij}(t), 1 \leq i,j \leq N$, by $\Gamma_{ij}(t)dt = (U(t)^\dagger d\Xi(t) U(t))_{ij} (U(t)^\dagger d\Xi(t) U(t))_{ji}$, where $d\Xi(t) = (d\xi_{ij})_{1\leq i,j \leq N}$. We denote by $\mathbf{1}(\omega)$ the indicator function: $\mathbf{1}(\omega) = 1$ if the condition ω is satisfied, and $\mathbf{1}(\omega) = 0$ otherwise. The following theorem is proved for the stochastic process of eigenvalues $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t))$.

Theorem 1 *Assume that $\xi_{ij}(t), 1 \leq i < j \leq N$ are continuous semimartingales. The process $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_N(t))$ satisfies the stochastic differential equations*

$$d\lambda_i(t) = dM_i(t) + dJ_i(t), \quad 1 \leq i \leq N,$$

where $M_i(t)$ is the martingale with quadratic variation $\langle M_i \rangle_t = \int_0^t \Gamma_{ii}(s) ds$ and $J_i(t)$ is the process with finite variation given by

$$dJ_i(t) = \sum_{j=1}^{N} \frac{1}{\lambda_i(t) - \lambda_j(t)} \mathbf{1}(\lambda_i(t) \neq \lambda_j(t)) \Gamma_{ij}(t) dt + d\Upsilon_i(t)$$

where $d\Upsilon_i(t)$ is the finite-variation part of $(U(t)^\dagger d\Xi(t) U(t))_{ii}$.

Since this theorem is obtained by simple generalization of Theorem 1 in Bru [6], we call it Bru's theorem here. A key point to derive the theorem is applying the Ito rule for differentiating the product of matrix-valued semimartingales: If X and Y are $N \times N$ matrices with semimartingale elements, then

$$d(X^\dagger Y) = (dX)^\dagger Y + X^\dagger (dY) + (dX)^\dagger (dY).$$

B Four Basic Examples

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and assume $\nu \in \mathbb{N}$. Let $B_{ij}(t)$, $\widetilde{B}_{ij}(t)$, $1 \leq i \leq N+\nu, 1 \leq j \leq N$ be independent one-dimensional standard Brownian motions. For $1 \leq i,j \leq N$ we set

$$s_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} B_{ij}(t), & \text{if } i < j, \\ B_{ii}(t), & \text{if } i = j, \\ \frac{1}{\sqrt{2}} B_{ji}(t), & \text{if } i > j, \end{cases} \quad \text{and} \quad a_{ij}(t) = \begin{cases} \frac{1}{\sqrt{2}} \widetilde{B}_{ij}(t), & \text{if } i < j, \\ 0, & \text{if } i = j, \\ -\frac{1}{\sqrt{2}} \widetilde{B}_{ji}(t), & \text{if } i > j. \end{cases}$$

Here we show four basic examples of hermitian matrix-valued processes and applications of Theorem 1.

(i) The first example of hermitian matrix-valued process is defined by

$$\Xi(t) = (\xi_{ij}(t))_{1 \leq i,j \leq N} = (s_{ij}(t) + \sqrt{-1} a_{ij}(t))_{1 \leq i,j \leq N}, \quad t \in [0, \infty).$$

By definition $d\xi_{ij}(t) d\xi_{k\ell}(t) = \delta_{i\ell} \delta_{jk} dt$, $1 \leq i,j,k,\ell \leq N$, and thus $\Gamma_{ij}(t) = 1$. Therefore $\boldsymbol{\lambda}(t)$ solves the equations of Dyson's Brownian motion model (1) with $\beta = 2$.

(ii) The second example is given by

$$\Xi(t) = (s_{ij}(t))_{1 \leq i,j \leq N} \in \mathcal{S}(N), \quad t \in [0, \infty).$$

In this case $d\xi_{ij}(t) d\xi_{k\ell}(t) = (\delta_{i\ell}\delta_{jk} + \delta_{ik}\delta_{j\ell}) dt/2$, $1 \leq i,j,k,\ell \leq N$, and thus $\Gamma_{ij}(t)dt = (1 + \delta_{ij}) dt/2$, $1 \leq i,j \leq N$. Then $\boldsymbol{\lambda}(t)$ solves (1) with $\beta = 1$.

(iii) We consider an $(N+\nu) \times N$ matrix-valued process by $M(t) = (B_{ij}(t) + \sqrt{-1} \widetilde{B}_{ij})_{1 \leq i \leq N+\nu, 1 \leq j \leq N}$ and define the $N \times N$ hermitian matrix-valued process by

$$\Xi(t) = M(t)^\dagger M(t), \quad t \in [0, \infty). \tag{7}$$

Since the matrix $\Xi(t)$ is positive definite, the eigenvalues are nonnegative. By definition we see that the finite-variation part of $d\xi_{ij}(t)$ is $2(N+\nu)\delta_{ij}dt$ and $d\xi_{ij}(t)d\xi_{k\ell}(t) = 2(\xi_{i\ell}(t)\delta_{jk} + \xi_{kj}(t)\delta_{i\ell})dt$, $1 \le i,j,k,\ell \le N$, which imply that $d\Upsilon_i(t) = 2(N+\nu)dt$ and $\Gamma_{ij}(t) = 2(\lambda_i(t)+\lambda_j(t))$, $1 \le i,j \le N$. Since $\langle M_i \rangle_t = \int_0^t 4\lambda_i(s)ds$, the stochastic differential equations for $\boldsymbol{\lambda}(t)$ are given by

$$d\lambda_i(t) = 2\sqrt{\lambda_i(t)}dB_i(t) + \beta \left\{ (N+\nu) + \sum_{1 \le j \le N: j \ne i} \frac{\lambda_i(t)+\lambda_j(t)}{\lambda_i(t)-\lambda_j(t)} \right\} dt, \quad 1 \le i \le N, \tag{8}$$

with $\beta = 2$.

(iv) Set $B(t) = (B_{ij}(t))_{1 \le i \le N+\nu, 1 \le j \le N}$ and define

$$\Xi(t) = B(t)^T B(t) \in \mathcal{S}(N), \quad t \in [0,\infty). \tag{9}$$

We see that the finite-variation part of $d\xi_{ij}(t)$ is $(N+\nu)\delta_{ij}dt$ and $d\xi_{ij}(t)d\xi_{k\ell}(t) = (\xi_{ik}(t)\delta_{j\ell} + \xi_{i\ell}(t)\delta_{jk} + \xi_{jk}(t)\delta_{i\ell} + \xi_{j\ell}(t)\delta_{ik})dt$, $1 \le i,j,k,\ell \le N$. Then $d\Upsilon_i(t) = (N+\nu)dt$ and $\Gamma_{ij}(t) = (\lambda_i(t)+\lambda_j(t))(1+\delta_{ij})$, $1 \le i,j \le N$. The equations for $\boldsymbol{\lambda}(t)$ are given by (8) with $\beta = 1$.

The process (9) was called the Wishart process and studied as matrix generalization of squared Bessel process by Bru [7]. König and O'Connell [36] called the process (7) the Laguerre process and studied its eigenvalue process (8) with $\beta = 2$.

C Relation with the standard and chiral random matrix theories

Here we assume that $B_{ij}(0) = \widetilde{B}_{ij}(0) = 0$ for all $1 \le i \le N+\nu, 1 \le j \le N$, and thus the initial distribution of $\Xi(t)$ is the pointmass on an $N \times N$ zero matrix O; $\mu(\Xi \in \cdot; 0) = \delta_O$. In this case the distributions of $\Xi(t)$'s are related with those studies in the standard (Wigner-Dyson) random matrix theory [40] and the chiral random matrix theory [54, 53, 27, 51].

(i) *Example (i) and GUE.* For GUE with variance $\sigma^2 = t$ of random matrices in the space $\mathcal{H}(N) \cong \mathbb{R}^{d[\mathrm{A}]}$ with $d[\mathrm{A}] = N^2$, the probability density of eigenvalues $\boldsymbol{\lambda}$ in the condition (6) is given as [40]

$$q^{\mathrm{GUE}}(\boldsymbol{\lambda}; t) = \frac{t^{-d[\mathrm{A}]/2}}{C[\mathrm{A}]} \exp\left\{ -\frac{|\boldsymbol{\lambda}|^2}{2t} \right\} h^{\mathrm{A}}(\boldsymbol{\lambda})^2,$$

where $|\boldsymbol{\lambda}|^2 = \sum_{i=1}^N \lambda_i^2$. For (1) with $\beta = 2$, $p^{\mathrm{A}}(0, \mathbf{0}; t, \boldsymbol{\lambda}) = q^{\mathrm{GUE}}(\boldsymbol{\lambda}; t)$, $t > 0$.

(ii) *Example (ii) and GOE.* The probability density of eigenvalues $\boldsymbol{\lambda}$ with the condition (6) is given as [40]

$$q^{\mathrm{GOE}}(\boldsymbol{\lambda}; t) = \frac{t^{-d[\mathrm{A}']/2}}{C[\mathrm{A}']} \exp\left\{ -\frac{|\boldsymbol{\lambda}|^2}{2t} \right\} h^{\mathrm{A}}(\boldsymbol{\lambda})$$

for GOE with variance $\sigma^2 = t$ in $\mathcal{S}(N) \cong \mathbb{R}^{d[\mathrm{A}']}$, $d[\mathrm{A}'] = N(N+1)/2$. If we denote by $p^{\mathrm{A}'}(s, \boldsymbol{\lambda}; t, \boldsymbol{\lambda}')$ the transition probability density of the process (1) with $\beta = 1$ from $\boldsymbol{\lambda}$ at time s to $\boldsymbol{\lambda}'$ at time $t (> s)$, then $p^{\mathrm{A}'}(0, \mathbf{0}; t, \boldsymbol{\lambda}) = q^{\mathrm{GOE}}(\boldsymbol{\lambda}; t)$, $t > 0$.

(iii) *Example (iii) and chiral GUE.* We denote by $\mathcal{M}(N+\nu, N; \mathbb{C})$ and $\mathcal{M}(N+\nu, N; \mathbb{R})$ the spaces of $(N+\nu) \times N$ complex and real matrices, respectively. We see that $\mathcal{M}(N+\nu, N; \mathbb{C}) \cong \mathbb{R}^{2N(N+\nu)}$ and write its volume element as $\mathcal{V}(dM), M \in \mathcal{M}(N+\nu, N; \mathbb{C})$. The chiral Gaussian unitary ensemble (chGUE) with variance t is the ensemble of matrices $M \in \mathcal{M}(N+\nu, N; \mathbb{C})$ with the probability density

$$\mu_\nu^{\mathrm{chGUE}}(M; t) = \frac{t^{-N(N+\nu)/2}}{(2\pi)^{N(N+\nu)}} \exp\left\{ -\frac{1}{2t} \mathrm{Tr} M^\dagger M \right\} \tag{10}$$

with respect to $\mathcal{V}(dM)$. It is known [24] that any matrix $M \in \mathcal{M}(N+\nu, N; \mathbb{C})$ has family of pairs $(U, V), U \in \mathrm{U}(N+\nu), V \in \mathrm{U}(N)$, which transform M as $M = U^\dagger K V$, where $K \in \mathcal{M}(N+\nu, N; \mathbb{R})$ is in the form

$$K = \begin{pmatrix} \widehat{K} \\ O \end{pmatrix} \quad \text{with} \quad \widehat{K} = \mathrm{diag}\{\kappa_1, \kappa_2, \cdots, \kappa_N\}, \quad \kappa_i \geq 0, 1 \leq i \leq N,$$

and the $\nu \times N$ zero matrix O. We assume that U and V are chosen so that

$$0 \leq \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_N. \tag{11}$$

The matrices (U, K, V) can be regarded as "polar coordinates" in the space $\mathcal{M}(N+\nu, N; \mathbb{C})$. We have $M^\dagger M = V^\dagger \Lambda V$, where $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ with the relations $\lambda_i = \kappa_i^2, 1 \leq i \leq N$. Then $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_N)$ is a set of nonnegative square roots of the eigenvalues of $M^\dagger M$. Let $d\mu(U, V)$ be the Haar measure of the space $\mathrm{U}(N+\nu) \times \mathrm{U}(N)$ normalized as $\int_{\mathrm{U}(N+\nu)\times \mathrm{U}(N)} d\mu(U, V) = 1$ and $d\boldsymbol{\kappa} = \prod_{i=1}^N d\kappa_i$. Then we can show that

$$\mathcal{V}(dM) = \frac{(2\pi)^{N(N+\nu)}}{C_\nu} h^{((2\nu+1)/2)}(\boldsymbol{\kappa})^2 d\boldsymbol{\kappa} d\mu(U, V), \tag{12}$$

where $C_\nu = 2^{N(N+\nu-1)} \prod_{i=1}^N \{\Gamma(i)\Gamma(i+\nu)\}$ and

$$h^{(\alpha)}(\boldsymbol{\kappa}) = \prod_{1 \leq i < j \leq N} (\kappa_j^2 - \kappa_i^2) \prod_{k=1}^N \kappa_k^\alpha.$$

For any pair of unitary matrices $U \in \mathrm{U}(N+\nu)$ and $V \in \mathrm{U}(N)$, the probability $\mu_\nu^{\mathrm{chGUE}}(M; t)\mathcal{V}(dM)$ is invariant under the automorphism $M \to U^\dagger M V$. By integrating over $d\mu(U, V)$, we obtain the probability density of $\boldsymbol{\kappa}$ with the condition (11) as [54, 53, 27, 51]

$$q_\nu^{\mathrm{chGUE}}(\boldsymbol{\kappa}; t) = \frac{t^{-N(N+\nu)}}{C_\nu} \exp\left\{ -\frac{|\boldsymbol{\kappa}|^2}{2t} \right\} h^{((2\nu+1)/2)}(\boldsymbol{\kappa})^2.$$

König and O'Connell [36] studied the process (8) with $\beta = 2$ as a multivariate version of squared Bessel process. Here we consider the multivariate version of Bessel process by extracting the square roots of eigenvalues $\lambda_i(t) \geq 0$ of $\Xi(t) = M(t)^\dagger M(t)$. Setting $\kappa_i(t) = \sqrt{\lambda_i(t)} \geq 0, 1 \leq i \leq N$ in (8) with $\beta = 2$ and applying the Ito rule for differentials, we find that $\boldsymbol{\kappa}(t)$ solves the stochastic differential equations

$$dZ_i(t) = dB_i(t) + \frac{\beta}{2} \left[\frac{\gamma}{Z_i(t)} + \sum_{j:j\neq i} \left\{ \frac{1}{Z_i(t) - Z_j(t)} + \frac{1}{Z_i(t) + Z_j(t)} \right\} \right] dt, \quad 1 \leq i \leq N, \tag{13}$$

with $(\beta, \gamma) = (2, (2\nu+1)/2)$. If we denote the transition probability density of this process by $p^{(\nu)}(s, \cdot; t, \cdot)$ for $0 \leq s < t < \infty$, then

$$p^{(\nu)}(0, \mathbf{0}; t, \boldsymbol{\kappa}) = q_\nu^{\mathrm{chGUE}}(\boldsymbol{\kappa}; t), \quad t > 0. \tag{14}$$

(iv) *Example (iv) and chiral GOE.* We can see $\mathcal{M}(N+\nu, N; \mathbb{R}) \cong \mathbb{R}^{N(N+\nu)}$. The chiral Gaussian orthogonal ensemble (chGOE) with variance t is the ensemble of matrices $B \in \mathcal{M}(N+\nu, N; \mathbb{R}) \subset \mathcal{M}(N+\nu, N; \mathbb{C})$ with the probability density

$$\mu_\nu^{\mathrm{chGOE}}(B; t) = \frac{t^{-N(N+\nu)/2}}{(2\pi)^{N(N+\nu)/2}} \exp\left\{ -\frac{1}{2t} \mathrm{Tr} B^T B \right\} \tag{15}$$

with respect to the volume element $\mathcal{V}'(dB)$ of $\mathcal{M}(N+\nu, N; \mathbb{R})$. We can show that

$$\mathcal{V}'(dB) = \frac{(2\pi)^{N(N+\nu)/2}}{C_{\nu,\nu+1}} h^{(\nu)}(\boldsymbol{\kappa}) d\boldsymbol{\kappa} d\mu(U, V), \tag{16}$$

where $d\mu(U,V)$ is the normalized Haar measure of the space $\mathrm{O}(N+\nu)\times\mathrm{O}(N)$ and we have used the notation $C_{\nu,\kappa} = 2^{N(N+2\nu-\kappa-1)/2}\pi^{-N/2}\prod_{i=1}^{N}\{\Gamma(i/2)\Gamma((i+2\nu+1-\kappa)/2)\}$ and thus $C_{\nu,\nu+1} = 2^{N(N+\nu-2)/2}\pi^{-N/2}$ $\prod_{i=1}^{N}\{\Gamma(i/2)\Gamma((i+\nu)/2)\}$. The probability density of $\boldsymbol{\kappa}$ with (11) is given as [54, 53, 27, 51]

$$q_\nu^{\mathrm{chGOE}}(\boldsymbol{\kappa};t) = \frac{t^{-N(N+\nu)/2}}{C_{\nu,\nu+1}}\exp\left\{-\frac{|\boldsymbol{\kappa}|^2}{2t}\right\}h^{(\nu)}(\boldsymbol{\kappa}).$$

By setting $\kappa_i(t)=\sqrt{\lambda_i(t)}, 1\le i\le N$ in (8) with $\beta=1$, we can show that $\boldsymbol{\kappa}(t)=(\kappa_1(t),\kappa_2(t),\cdots,\kappa_N(t))$ solves (13) with $(\beta,\gamma)=(1,\nu)$. If we denote the transition probability density of this process $\boldsymbol{\kappa}(t)$ by $p^{(\nu)'}(s,\cdot\,;t,\cdot\,)$ for $0\le s<t<\infty$, then $p^{(\nu)'}(0,\mathbf{0};t,\boldsymbol{\kappa})=q_\nu^{\mathrm{chGOE}}(\boldsymbol{\kappa};t)$, $t>0$.

III HERMITIAN MATRIX-VALUED PROCESSES WITH ADDITIONAL SYMMETRIES

A Subspaces of unitary and hermitian matrices

The Pauli spin matrices are defined as

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-\sqrt{-1}\\ \sqrt{-1}&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

which satisfy the algebra $\sigma_\mu^2=I_2, \mu=1,2,3$, and $\sigma_\mu\sigma_\rho=\sqrt{-1}\sum_{\omega=1}^{3}\varepsilon_{\mu\rho\omega}\sigma_\omega$ for $1\le\mu\ne\rho\le 3$, where I_N denotes the $N\times N$ unit matrix and $\varepsilon_{\mu\rho\omega}$ the totally antisymmetric unit tensor. They give the infinitesimal generators $\{X_\mu\}$ of SU(2) by $X_\mu=\sqrt{-1}\sigma_\mu/2$. For $N\ge 2$, define the $2N\times 2N$ matrices $\Sigma_\mu=I_N\otimes\sigma_\mu, \mu=1,2,3$. The matrices $\{\Sigma_\mu\}$ satisfy the same algebra as $\{\sigma_\mu\}$. We will use σ_0 to represent I_2.

We introduce six spaces of matrices as subspaces of $\mathcal{H}(2N)$,

$$\mathcal{H}_{\mu\pm}(2N)=\{H\in\mathcal{H}(2N): H^T\Sigma_\mu=\pm\Sigma_\mu H\},\quad \mu=1,2,3.$$

It is easy to see that $\mathcal{H}_{3+}(2N)=\mathcal{S}(2N)$ and $\mathcal{H}_{3-}(2N)=\sqrt{-1}\mathcal{A}(2N)$. Since we have already studied the matrix-valued process in $\mathcal{S}(N)$ as the example (ii) in Sec.II.B, we will consider here the five subspaces of $\mathcal{H}(2N)$; $\sqrt{-1}\mathcal{A}(2N)$ and $\{\mathcal{H}_{\mu\sigma}(2N)\}$ with $\mu=1,2,\sigma=\pm$. We also introduce the three subspaces of $\mathrm{U}(2N)$:

$$\begin{aligned}\mathrm{U}_0(2N) &= \{U\in\mathrm{U}(2N): U^TU=\Sigma_1\},\\ \mathrm{U}_\mu(2N) &= \{U\in\mathrm{U}(2N): U^T\Sigma_\mu U=\Sigma_\mu^T\},\quad \mu=1,2.\end{aligned}$$

The conditions imply that these subspaces, $\mathcal{H}_{\mu\sigma}(2N)$ and $\mathrm{U}_\mu(2N)$, have additional symmetries compared to $\mathcal{H}(2N)$ and $\mathrm{U}(2N)$. Concerning the eigenvalues and eigenvectors of the hermitian matrices, the following lemma may be easily proved.

Lemma 2 *Assume that* Ω *denotes a diagonal matrix in the form* $\mathrm{diag}\{\omega_1,\omega_2,\cdots,\omega_N\}$ *with* $\omega_1\le\omega_2\le\cdots\le\omega_N$.

(i) *Any* $H\in\sqrt{-1}\mathcal{A}(2N)$ *can be diagonalized by* $U\in\mathrm{U}_0(2N)$ *as* $U^\dagger HU=\Omega\otimes\sigma_3$.

(ii) *For* $\mu=1,2$ *any* $H\in\mathcal{H}_{\mu+}(2N)$ *can be diagonalized by* $U\in\mathrm{U}_\mu(2N)$ *as* $U^\dagger HU=\Omega\otimes\sigma_0$.

(iii) *For* $\mu=1,2$ *any* $H\in\mathcal{H}_{\mu-}(2N)$ *can be diagonalized by* $U\in\mathrm{U}_\mu(2N)$ *as* $U^\dagger HU=\Omega\otimes\sigma_3$.

Remark

(a) Observing the pairing of eigenvalues in a way, $(\omega_i,-\omega_i), 1\le i\le N$, for $\sqrt{-1}\mathcal{A}(2N)$ stated in Lemma 2 (i), the Gaussian random-matrix ensemble of antisymmetric hermitian matrices was discussed by Mehta in Section 3.4 of [40].

(b) The condition for $\mathrm{U}_2(2N)$ addition to the unitarity is equivalent with $J = UJU^T$, where

$$J = I_N \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\mathrm{U}_2(2N)$ forms the N-dimensional symplectic group. That is, $\mathrm{U}_2(2N) = \mathrm{Sp}(N, \mathbb{C}) \cap \mathrm{U}(2N)$. (It is called the unitary-symplectic group $\mathrm{USp}(2N)$ in [19].) The matrices $H \in \mathcal{H}_{2+}(2N)$ are said to be *self-dual hermitian matrices* in the random matrix theory [40]. The pairwise degeneracy stated in Lemma 2 (ii) for $\mathcal{H}_{2+}(2N)$ is known as the *Kramers doublet* in the quantum mechanics.

(c) The condition for $\mathcal{H}_{2-}(2N)$ addition to hermiticity is rewritten as $H^T J + JH = 0$, which means that $H \in \mathcal{H}_{2-}(2N)$ satisfies the symplectic Lie algebra (see for example [18]), that is, $\mathcal{H}_{2-}(2N) = \mathfrak{sp}(2N, \mathbb{C}) \cap \mathcal{H}(2N)$. Similarly, we can see $\mathcal{H}_{1-}(2N) = \mathfrak{so}(2N, \mathbb{C}) \cap \mathcal{H}(2N)$, where $\mathfrak{so}(2N, \mathbb{C})$ denotes the orthogonal Lie algebra. We can also see that $\mathrm{U}_1(2N) = \mathrm{SO}(2N, \mathbb{C}) \cap \mathrm{U}(2N)$, where $\mathrm{SO}(2N, \mathbb{C})$ denotes the orthogonal Lie group.

(d) We can see that $\mathcal{H}_{\mu-}(2N) \cong \widehat{\mathcal{H}}_{\mu-}(2N), \mu = 1, 2$, where

$$\begin{aligned}
\widehat{\mathcal{H}}_{1-}(2N) &= \left\{ H = \begin{pmatrix} H_1 & A_2 \\ A_2^\dagger & -H_1^T \end{pmatrix} : H_1 \in \mathcal{H}(N), A_2 \in \mathcal{A}(N; \mathbb{C}) \right\}, \\
\widehat{\mathcal{H}}_{2-}(2N) &= \left\{ H = \begin{pmatrix} H_1 & A_2 \\ A_2^\dagger & -H_1^T \end{pmatrix} : H_1 \in \mathcal{H}(N), A_2 \in \mathcal{S}(N; \mathbb{C}) \right\},
\end{aligned}$$

where $\mathcal{S}(N; \mathbb{C})$ and $\mathcal{A}(N; \mathbb{C})$ denote the spaces of the $N \times N$ complex symmetric and complex anti-symmetric matrices, respectively. Altland and Zirnbauer studied $\widehat{\mathcal{H}}_{2-}(2N)$ and $\widehat{\mathcal{H}}_{1-}(2N)$ as the sets of the Hamiltonians in the Bogoliubov-de Gennes formalism for the BCS mean-field theory of superconductivity, where the pairing of positive and negative eigenvalues $(\omega_i, -\omega_i), 1 \le i \le N$, stated in Lemma 2 (iii) for $\mu = 1$ and 2 represents the particle-hole symmetry in the Bogoliubov-de Gennes theory. They called $\widehat{\mathcal{H}}_{2-}(2N)$ and $\widehat{\mathcal{H}}_{1-}(2N)$ the sets of hermitian matrices in the symmetry classes C and D [56, 1, 2], since $\mathfrak{sp}(2N, \mathbb{C}) = \mathrm{C}_N$ and $\mathfrak{so}(2N, \mathbb{C}) = \mathrm{D}_N$ in Cartan's notations (see [23]).

B Representation using Pauli matrices and application of Bru's theorem

Let $B_{ij}^\rho(t)$, $\widetilde{B}_{ij}^\rho(t)$, $0 \le \rho \le 3$, $1 \le i \le j \le N$ be independent one-dimensional standard Brownian motions starting from the origin. Put

$$s_{ij}^\rho(t) = \begin{cases} \dfrac{1}{\sqrt{2}} B_{ij}^\rho(t), & \text{if } i < j, \\[2ex] B_{ii}^\rho(t), & \text{if } i = j, \end{cases} \quad \text{and} \quad a_{ij}^\rho(t) = \begin{cases} \dfrac{1}{\sqrt{2}} \widetilde{B}_{ij}^\rho(t), & \text{if } i < j, \\[2ex] 0, & \text{if } i = j, \end{cases} \tag{17}$$

with $s_{ij}^\rho(t) = s_{ji}^\rho(t)$ and $a_{ij}^\rho(t) = -a_{ji}^\rho(t)$ for $i > j$ and define $s^\rho(t) = (s_{ij}^\rho(t))_{1 \le i,j \le N} \in \mathcal{S}(N)$, $t \in [0, \infty)$ and $a^\rho(t) = (a_{ij}^\rho(t))_{1 \le i,j \le N} \in \mathcal{A}(N), t \in [0, \infty)$, for $0 \le \rho \le 3$.

We can see that the hermitian matrix-valued process given as the first example (i) in Sec. II.B can be represented, if we double the size of matrix to $2N$, as $\Xi(t) = \sum_{\rho=0}^{3} \{(s^\rho(t) \otimes \sigma_\rho) + \sqrt{-1}(a^\rho(t) \otimes \sigma_\rho)\}$. By choosing four terms in the eight terms, we define the following four different types of $2N \times 2N$ hermitian matrix-valued processes:

$$\Xi_{\mu\sigma}(t) = \sum_{\rho=0}^{3} (\xi_{\mu\sigma}^\rho(t) \otimes \sigma_\rho) \in \mathcal{H}_{\mu\sigma}(2N) \quad \text{for} \quad \mu = 1, 2, \quad \sigma = \pm,$$

where

$$\xi_{\mu+}^\rho(t) = \begin{cases} s^\rho(t) & \text{if } \mu = 1, \quad \rho \neq 3 \quad \text{or} \quad \mu = 2, \quad \rho = 0, \\[2ex] \sqrt{-1} a^\rho(t) & \text{if } \mu = 1, \quad \rho = 3 \quad \text{or} \quad \mu = 2, \quad \rho \neq 0, \end{cases}$$

$$\xi^{\rho}_{\mu-}(t) = \begin{cases} \sqrt{-1}a^{\rho}(t) & \text{if } \mu = 1, \quad \rho \neq 3 \quad \text{or} \quad \mu = 2, \quad \rho = 0, \\ \\ s^{\rho}(t) & \text{if } \mu = 1, \quad \rho = 3 \quad \text{or} \quad \mu = 2, \quad \rho \neq 0. \end{cases}$$

We apply Theorem 1 to the five processes, $\sqrt{-1}\mathcal{A}(2N)$ and $\{\Xi_{\mu\sigma}(t)\}$ with $\mu = 1, 2, \sigma = \pm$. The results are listed below.

(a) $\sqrt{-1}\mathcal{A}(2N)$: Since $\Gamma_{ij}(t) = \{1 - ((\Sigma_1)_{ij})^2\}/2$, $1 \leq i, j \leq 2N$, the equations of nonnegative eigenvalues are

$$d\omega_i(t) = \frac{1}{\sqrt{2}} dB_i(t) + \frac{1}{2} \sum_{j:1 \leq j \leq N, j \neq i} \left\{ \frac{1}{\omega_i(t) - \omega_j(t)} + \frac{1}{\omega_i(t) + \omega_j(t)} \right\} dt, \quad 1 \leq i \leq N.$$

By changing the time unit as $t \to 2t$, this equation can be identified with (13) with $(\beta, \gamma) = (2, 0)$.

(b) $\mathcal{H}_{1+}(2N)$: Since $\Gamma_{ij}(t) = \left\{1 + ((\Sigma_1)_{ij})^2\right\}$, $1 \leq i, j \leq 2N$, the distinct eigenvalues solve Dyson's Brownian motion model (1) with $\beta = 4$.

(c) $\mathcal{H}_{1-}(2N)$: We see $\Gamma_{ij}(t) = \left\{1 - ((\Sigma_1)_{ij})^2\right\}$, $1 \leq i, j \leq 2N$. Then the nonnegative eigenvalues solve the equations (13) with $(\beta, \gamma) = (2, 0)$.

(d) $\mathcal{H}_{2+}(2N)$: Since $\Gamma_{ij}(t) = \left\{1 + ((\Sigma_2)_{ij})^2\right\}$, $1 \leq i, j \leq 2N$, the distinct eigenvalues solve the equations (1) with $\beta = 4$

(e) $\mathcal{H}_{2-}(2N)$: We can see $\Gamma_{ij}(t) = \left\{1 - ((\Sigma_2)_{ij})^2\right\}$, $1 \leq i, j \leq 2N$. Then the nonnegative eigenvalues solve the equation (13) with $(\beta, \gamma) = (2, 1)$.

C Relation with standard and nonstandard random matrix theories

(i) The eigenvalues of any matrix in the space $\mathcal{H}_{2+}(2N) \cong \mathbb{R}^{d[\mathrm{A}'']}$ with $d[\mathrm{A}''] = N(2N-1)$ are pairwise degenerated (the Kramers doublets) as $\boldsymbol{\lambda} = (\omega_1, \omega_1, \omega_2, \omega_2, \cdots, \omega_N, \omega_N)$. We assume that the N distinct eigenvalues are always arranged in the increasing order $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_N$. For GSE with variance t, the probability density of the N distinct eigenvalues in this ordering is given by [40]

$$q^{\mathrm{GSE}}(\boldsymbol{\omega}; t) = \frac{t^{-d[\mathrm{A}'']/2}}{C[\mathrm{A}'']} \exp\left\{ -\frac{|\boldsymbol{\omega}|^2}{2t} \right\} h^{\mathrm{A}}(\boldsymbol{\omega})^4,$$

where $C[\mathrm{A}''] = (2\pi)^{N/2} \prod_{i=1}^{N} \Gamma(2i)$. If we denote the transition probability density of the process (1) with $\beta = 4$ by $p^{\mathrm{A}''}(s, \cdot; t, \cdot)$ for $0 \leq s < t < \infty$, then $p^{\mathrm{A}''}(0, \mathbf{0}; t, \boldsymbol{\omega}) = q^{\mathrm{GSE}}(\boldsymbol{\omega}; t)$, $t > 0$.

(ii) We can see that $\mathcal{H}_{2-}(2N) \cong \mathbb{R}^{d[\mathrm{C}]}$ and $\mathcal{H}_{1-}(2N) \cong \mathbb{R}^{d[\mathrm{D}]}$ with $d[\mathrm{C}] = N(2N+1)$ and $d[\mathrm{D}] = N(2N-1)$. The probability densities of the processes $\Xi_{2-}(t)$ and $\Xi_{1-}(t)$ with respect to the volume elements $\mathcal{V}(dH)$ of $\mathcal{H}_{2-}(2N)$ and $\mathcal{V}'(dH)$ of $\mathcal{H}_{1-}(2N)$ are given by

$$\mu^{\mathrm{C}}(H; t) = \frac{t^{-d[\mathrm{C}]/2}}{c[\mathrm{C}]} \exp\left\{ -\frac{1}{4t} \mathrm{Tr}\, H^2 \right\}, \quad \mu^{\mathrm{D}}(H; t) = \frac{t^{-d[\mathrm{D}]/2}}{c[\mathrm{D}]} \exp\left\{ -\frac{1}{4t} \mathrm{Tr}\, H^2 \right\},$$

where $c[\mathrm{C}] = 2^{3N/2} \pi^{N(2N+1)/2}$ and $c[\mathrm{D}] = 2^{N/2} \pi^{N(2N-1)/2}$, respectively. As stated in Lemma 2 (iii), the eigenvalues are in the form $\boldsymbol{\lambda}(t) = (\omega_1(t), -\omega_1(t), \omega_2(t), -\omega_2(t), \cdots, \omega_N(t), -\omega_N(t))$. We will assume that

$$0 \leq \omega_1 \leq \omega_2 \leq \cdots \leq \omega_N. \tag{18}$$

Then we have the expressions for volume elements

$$\mathcal{V}(dH) = \frac{c[\mathrm{C}]}{C[\mathrm{C}]} h^{\mathrm{C}}(\boldsymbol{\omega})^2 d\boldsymbol{\omega} dU, \quad \mathcal{V}'(dH) = \frac{c[\mathrm{D}]}{C[\mathrm{D}]} h^{\mathrm{D}}(\boldsymbol{\omega})^2 d\boldsymbol{\omega} dU', \tag{19}$$

where dU and dU' denote the Haar measures of $\mathrm{U}_2(2N)$ and $\mathrm{U}_1(2N)$, respectively, normalized as $\int_{\mathrm{U}_2(2N)} dU = 1$ and $\int_{\mathrm{U}_1(2N)} dU' = 1$. Here $C[\mathrm{C}] = C_{1/2} = (\pi/2)^{N/2} \prod_{i=1}^N \Gamma(2i)$ and $C[\mathrm{D}] = C_{-1/2} = (\pi/2)^{N/2} \prod_{i=1}^N \Gamma(2i-1)$, and $h^{\mathrm{C}}(\boldsymbol{\omega}) \equiv h^{(1)}(\boldsymbol{\omega}), h^{\mathrm{D}}(\boldsymbol{\omega}) \equiv h^{(0)}(\boldsymbol{\omega})$. At each time $t > 0$, for any $U \in \mathrm{U}_2(2N)$, the probability $\mu^{\mathrm{C}}(H;t)\mathcal{V}(dH)$ is invariant under the automorphism $H \to U^\dagger H U$ for $H \in \mathcal{H}_{2-}(2N)$, and for any $U' \in \mathrm{U}_1(2N)$, $\mu^{\mathrm{D}}(H;t)\mathcal{V}'(dH)$ is invariant under the automorphism $H \to U'^\dagger H U'$ for $H \in \mathcal{H}_{1-}(2N)$. Altland and Zirnbauer named these two Gaussian random-matrix ensembles the classes C and D, respectively (see Remark (d) in Sec.III.A) [1, 2, 56]. The probability densities of the N nonnegative eigenvalues with the condition (18) are then obtained as

$$q^\sharp(\boldsymbol{\omega};t) = \frac{t^{-d[\sharp]/2}}{C[\sharp]} \exp\left\{-\frac{|\boldsymbol{\omega}|^2}{2t}\right\} h^\sharp(\boldsymbol{\omega})^2 \quad \text{for} \quad \sharp = \mathrm{C, D}.$$

If we denote the transition probability densities of the processes (13) with $(\beta,\gamma) = (2,1)$ and with $(\beta,\gamma) = (2,0)$ by $p^{\mathrm{C}}(s, \cdot\,; t, \cdot\,)$ and $p^{\mathrm{D}}(s, \cdot\,; t, \cdot\,)$ for $0 \le s < t < \infty$, respectively, then

$$p^\sharp(0, \mathbf{0}; t, \boldsymbol{\omega}) = q^\sharp(\boldsymbol{\omega};t), \quad t > 0 \quad \text{for} \quad \sharp = \mathrm{C, D}. \tag{20}$$

D Real symmetric matrix-valued processes

Here after, we denote the hermitian matrix-valued processes $\Xi_{2-}(t)$ and $\Xi_{1-}(t)$ by $\Xi^{\mathrm{C}}(t)$ and $\Xi^{\mathrm{D}}(t)$, respectively. They are given by

$$\begin{aligned} \Xi^{\mathrm{C}}(t) &= \sqrt{-1}a^0(t) \otimes \sigma_0 + s^1(t) \otimes \sigma_1 + s^2(t) \otimes \sigma_2 + s^3(t) \otimes \sigma_3, \\ \Xi^{\mathrm{D}}(t) &= \sqrt{-1}a^0(t) \otimes \sigma_0 + \sqrt{-1}a^1(t) \otimes \sigma_1 + \sqrt{-1}a^2(t) \otimes \sigma_2 + s^3(t) \otimes \sigma_3. \end{aligned} \tag{21}$$

Since $\sigma_\rho, \rho = 0,1,3$, are real matrices and σ_2 is a pure imaginary matrix, if we define the processes as

$$\Xi^{\mathrm{C}'}(t) = s^1(t) \otimes \sigma_1 + s^3(t) \otimes \sigma_3, \quad \Xi^{\mathrm{D}'}(t) = \sqrt{-1}a^2(t) \otimes \sigma_2 + s^3(t) \otimes \sigma_3, \tag{22}$$

then $\Xi^{\mathrm{C}'}(t) \in \mathcal{S}_{2-}(2N)$ and $\Xi^{\mathrm{D}'}(t) \in \mathcal{S}_{1-}(2N)$, where $\mathcal{S}_{2-}(2N) \equiv \{S \in \mathcal{S}(2N) : S^T\Sigma_2 = -\Sigma_2 S\} \cong \mathbb{R}^{d[\mathrm{C}']}$ and $\mathcal{S}_{1-}(2N) \equiv \{S \in \mathcal{S}(2N) : S^T\Sigma_1 = -\Sigma_1 S\} \cong \mathbb{R}^{d[\mathrm{D}']}$ with $d[\mathrm{C}'] = N(N+1)$ and $d[\mathrm{D}'] = N^2$. The probability densities of $\Xi^{\mathrm{C}'}(t)$ and $\Xi^{\mathrm{D}'}(t)$ are given by

$$\mu^{\mathrm{C}'}(S;t) = \frac{t^{-d[\mathrm{C}']/2}}{c[\mathrm{C}']} \exp\left\{-\frac{1}{4t}\mathrm{Tr}\, S^2\right\}, \quad \mu^{\mathrm{D}'}(S;t) = \frac{t^{-d[\mathrm{D}']/2}}{c[\mathrm{D}']} \exp\left\{-\frac{1}{4t}\mathrm{Tr}\, S^2\right\}$$

with $c[\mathrm{C}'] = 2^N \pi^{N(N+1)/2}$ and $c[\mathrm{D}'] = 2^{N/2}\pi^{N^2/2}$, respectively. Set $\mathrm{O}_2(2N) = \mathrm{O}(2N) \cap \mathrm{Sp}(2N;\mathbb{R})$ and $\mathrm{O}_1(2N) = \mathrm{O}(2N) \cap \mathrm{SO}(2N;\mathbb{R})$ and denote their normalized Haar measures by dV and dV', respectively The eigenvalues are in the form $\boldsymbol{\lambda}(t) = (\omega_1(t), -\omega_1(t), \omega_2(t), -\omega_2(t), \cdots, \omega_N(t), -\omega_N(t))$. Under the condition (18), we have the expressions for volume elements

$$\mathcal{V}(dS) = \frac{c[\mathrm{C}']}{C[\mathrm{C}']} h^{\mathrm{C}}(\boldsymbol{\omega}) d\boldsymbol{\omega} dV, \quad \mathcal{V}'(dS) = \frac{c[\mathrm{D}']}{C[\mathrm{D}']} h^{\mathrm{D}}(\boldsymbol{\omega}) d\boldsymbol{\omega} dV', \tag{23}$$

where $C[\mathrm{C}'] = C_{1/2,1} = \prod_{i=1}^N \Gamma(i)$ and $C[\mathrm{D}'] = C_{-1/2,0} = 2^{(N-2)/2}\Gamma(N/2)\prod_{i=1}^{N-1}\Gamma(i)$. The probability densities of the N nonnegative eigenvalues with (18) are given as

$$q^{\sharp'}(\boldsymbol{\omega};t) = \frac{t^{-d[\sharp']/2}}{C[\sharp']} \exp\left\{-\frac{|\boldsymbol{\omega}|^2}{2t}\right\} h^\sharp(\boldsymbol{\omega}) \quad \text{for} \quad \sharp = \mathrm{C, D}.$$

It is remarked that the random-matrix ensemble with the distributions $\mu^{\mathrm{C}'}(S;t)$, whose nonnegative eigenvalue distribution is given by $q^{\mathrm{C}'}(\boldsymbol{\omega};t)$, is the random-matrix ensemble in the symmetry class CI studied by Altland and Zirnbauer [1, 2, 56].

By applying Theorem 1, we can show that the nonnegative eigenvalues of $\Xi^{\mathrm{C}'}(t)$ solve the equations (13) with $(\beta,\gamma) = (1,1)$ and those of $\Xi^{\mathrm{D}'}(t)$ the equations (13) with $(\beta,\gamma) = (1,0)$. If we denote the transition probability densities of these processes by $p^{\mathrm{C}'}(s, \cdot\,; t, \cdot\,)$ and $p^{\mathrm{D}'}(s, \cdot\,; t, \cdot\,)$ for $0 \le s < t < \infty$, respectively, then $p^{\sharp'}(0, \mathbf{0}; t, \boldsymbol{\omega}) = q^{\sharp'}(\boldsymbol{\omega};t)$, $t > 0$ for $\sharp = \mathrm{C, D}$.

IV TEMPORALLY HOMOGENEOUS PROCESSES

Assume that $\nu > -1$, and we consider the process $\mathbf{Y}^{(\nu)}(t) = (Y_1^{(\nu)}(t), Y_2^{(\nu)}(t), \cdots, Y_N^{(\nu)}(t)), t \in [0, \infty)$, which solves the stochastic differential equations (13) with $(\beta, \gamma) = (2, (2\nu+1)/2)$, that is,

$$dY_i^{(\nu)}(t) = dB_i(t) + \left[\frac{2\nu+1}{2Y_i^{(\nu)}(t)} + \sum_{j:j\neq i} \left\{ \frac{1}{Y_i^{(\nu)}(t) - Y_j^{(\nu)}(t)} + \frac{1}{Y_i^{(\nu)}(t) + Y_j^{(\nu)}(t)} \right\} \right] dt, \tag{24}$$

$1 \leq i \leq N$. Remark that if $\nu = 1/2$ and $-1/2$, the equation is reduced to (13) with $(\beta, \gamma) = (2, 1)$ and $(\beta, \gamma) = (2, 0)$, respectively. The Kolmogorov backward equation (the Fokker-Planck equation) for (24) is

$$\frac{\partial}{\partial t} p^{(\nu)}(s, \mathbf{x}; t, \mathbf{y}) = \frac{1}{2} \Delta_{\mathbf{x}} p^{(\nu)}(s, \mathbf{x}; t, \mathbf{y}) + \mathbf{b}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} p^{(\nu)}(s, \mathbf{x}; t, \mathbf{y}),$$

where $\mathbf{b}(\mathbf{x}) = (b_1(\mathbf{x}), \cdots, b_N(\mathbf{x}))$ with $b_i(\mathbf{x}) = (\partial/\partial x_i) \ln h^{((2\nu+1)/2)}(\mathbf{x})$. By simple calculation, we can confirm the following.

Lemma 3 *Set*

$$f^{(\nu)}(t, \mathbf{y}|\mathbf{x}) = \det_{1 \leq i,j \leq N} \left[G^{(\nu)}(t, y_j|x_i) \right]. \tag{25}$$

Then the transition probability density $p^{(\nu)}(s, \mathbf{x}; t, \mathbf{y})$ from the state $\mathbf{x}$ at time s to the state $\mathbf{y}$ at time $t(> s)$ of the process (24) is given by

$$p^{(\nu)}(s, \mathbf{x}; t, \mathbf{y}) = \frac{1}{h^{(0)}(\mathbf{x})} f^{(\nu)}(t-s, \mathbf{y}|\mathbf{x}) h^{(0)}(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}. \tag{26}$$

Since $I_{1/2}(x) = (e^x - e^{-x})/\sqrt{2\pi x}$, $I_{-1/2}(x) = (e^x + e^{-x})/\sqrt{2\pi x}$, if we set

$$G^{\mathrm{C}}(t, y|x) = \frac{e^{-(y-x)^2/2t} - e^{-(y+x)^2/2t}}{\sqrt{2\pi t}}, \quad G^{\mathrm{D}}(t, y|x) = \frac{e^{-(y-x)^2/2t} + e^{-(y+x)^2/2t}}{\sqrt{2\pi t}}, \tag{27}$$

and $f^\sharp(t, \mathbf{y}|\mathbf{x}) = \det_{1 \leq i,j \leq N}[G^\sharp(t, y_j|x_i)]$, $\sharp =$ C, D, then

$$p^{(1/2)}(s, \mathbf{x}; t, \mathbf{y}) = \frac{f^{\mathrm{C}}(t-s, \mathbf{y}|\mathbf{x}) h^{\mathrm{C}}(\mathbf{y})}{h^{\mathrm{C}}(\mathbf{x})}, \quad p^{(-1/2)}(s, \mathbf{x}; t, \mathbf{y}) = \frac{f^{\mathrm{D}}(t-s, \mathbf{y}|\mathbf{x}) h^{\mathrm{D}}(\mathbf{y})}{h^{\mathrm{D}}(\mathbf{x})}. \tag{28}$$

The above implies the following. Let $\mathbb{W}_N^{\mathrm{C}} = \{\mathbf{x} \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N\}$ and $\mathbb{W}_N^{\mathrm{D}} = \{\mathbf{x} \in \mathbb{R}^N : |x_1| < x_2 < \cdots < x_N\}$. The former is the Weyl chamber of type C_N and the latter of type D_N [18]. Since h^{C} and h^{D} vanish at the boundaries of the Weyl chambers $\mathbb{W}_N^{\mathrm{C}}$ and $\mathbb{W}_N^{\mathrm{D}}$, respectively, (28) implies that the processes $\mathbf{Y}^{(1/2)}(t)$ and $\mathbf{Y}^{(-1/2)}(t)$ can be regarded as the N-dimensional absorbing Brownian motions in $\mathbb{W}_N^{\mathrm{C}}$ and in $\mathbb{W}_N^{\mathrm{D}}$, respectively. That is, if $\mathbf{Y}^{(1/2)}(0) \in \mathbb{W}_N^{\mathrm{C}}$ and $\mathbf{Y}^{(-1/2)}(0) \in \mathbb{W}_N^{\mathrm{D}}$, then $\mathbf{Y}^{(1/2)}(t) \in \mathbb{W}_N^{\mathrm{C}}$ and $\mathbf{Y}^{(-1/2)}(t) \in \mathbb{W}_N^{\mathrm{D}}$ for all $t > 0$ with probability 1. Moreover, we notice that (27) are the heat-kernels of the one-dimensional Brownian motion with an absorbing wall at the origin, and of the one-dimensional reflecting Brownian motion, respectively [48]. Then, we can also interpret the process $\mathbf{Y}^{(1/2)}(t)$ as the N-particle system of Brownian motions conditioned never to collide with each other nor with the wall at the origin in one-dimension [35], and the process $\mathbf{Y}^{(-1/2)}(t)$ as the N-particle system of reflecting Brownian motions conditioned never to collide with each other. For $\sharp =$C and D, define

$$\mathcal{N}^\sharp(t, \mathbf{x}) = \int_{\mathbb{W}_N^\sharp} d\mathbf{y}\, f^\sharp(t, \mathbf{y}|\mathbf{x}), \quad \mathbf{x} \in \mathbb{W}_N^\sharp. \tag{29}$$

$\mathcal{N}^{\mathrm{C}}(t, \mathbf{x})$ is the probability that N Brownian motions starting from $\mathbf{x} \in \mathbb{W}_N^{\mathrm{C}}$ does not collide with each other nor with the wall at the origin up to time t, and $\mathcal{N}^{\mathrm{D}}(t, \mathbf{x})$ is equal to the probability that N reflecting Brownian motions starting from $\mathbf{x} \in \mathbb{W}_N^{\mathrm{D}}$ does not collide with each other up to time t, respectively. We will show their long-time asymptotics in the next section. We can prove the following, which are consistent with (14) and (20).

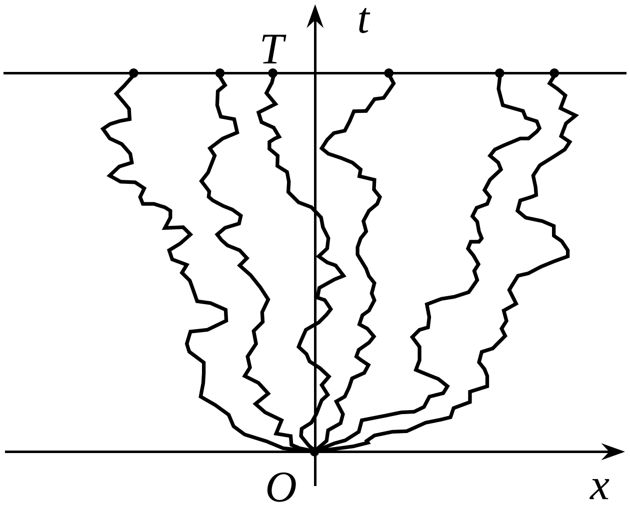

Figure 1: Process $\mathbf{X}(t), t \in [0,T]$, with $\mathbf{X}(0)=\mathbf{0}$ showing star topology.

Lemma 4 *For $\nu > -1$ with fixed $t \in (0,\infty)$, assume $\mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}$. Then*

$$\lim_{|\mathbf{x}|\to 0} p^{(\nu)}(0,\mathbf{x};t,\mathbf{y}) = \frac{t^{-N(N+\nu)}}{C_\nu} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h^{((2\nu+1)/2)}(\mathbf{y})^2. \tag{30}$$

In particular, if $\nu \in \mathbb{N}$,

$$\lim_{|\mathbf{x}|\to 0} p^{(\nu)}(0,\mathbf{x};t,\mathbf{y}) = q_\nu^{\mathrm{chGUE}}(\mathbf{y};t), \tag{31}$$

and

$$\lim_{|\mathbf{x}|\to 0} p^{(1/2)}(0,\mathbf{x};t,\mathbf{y}) = q^{\mathrm{C}}(\mathbf{y};t), \qquad \lim_{|\mathbf{x}|\to 0} p^{(-1/2)}(0,\mathbf{x};t,\mathbf{y}) = q^{\mathrm{D}}(\mathbf{y};t). \tag{32}$$

Proof. By definition (25) with (3), if $x_i > 0,\ 1 \le \forall i \le N$, $f^{(\nu)}(t,\mathbf{y}|\mathbf{x}) = (1/t^N)\prod_{k=1}^N (y_k^{\nu+1}/x_k^\nu)$ $e^{-(|\mathbf{x}|^2+|\mathbf{y}|^2)/2t} \det_{1\le i,j\le N}[I_\nu(x_i y_j/t)]$. We can use (A.2) in Appendix A by changing the variables $x_i \to x_i^2/2t$ and $y_j \to y_j^2/2t$ to evaluate $\det_{1\le i,j\le N}[I_\nu(x_i y_j/t)]$ and obtain the asymptotic form of $f^{(\nu)}(t,\mathbf{y}|\mathbf{x})$,

$$\begin{aligned} f^{(\nu)}(t,\mathbf{y}|\mathbf{x}) \quad &= \quad \frac{t^{-N(N+2\nu+1)/2}}{C_\nu} \prod_{1\le i<j\le N} \left\{ \left(\frac{x_j}{\sqrt{t}}\right)^2 - \left(\frac{x_i}{\sqrt{t}}\right)^2 \right\} \\ &\quad \times \prod_{1\le k<\ell\le N} (y_\ell^2 - y_k^2) \prod_{m=1}^N y_m^{2\nu+1} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \times \left(1+\mathcal{O}\left(\frac{|\mathbf{x}|}{\sqrt{t}}\right)\right) \end{aligned} \tag{33}$$

in $|\mathbf{x}|/\sqrt{t} \to 0$. Using this form in (26), the limit (30) is proved. ∎

V TEMPORALLY INHOMOGENEOUS PROCESSES

A Star topology

Using (2) the probability that the Brownian motion started at $\mathbf{x} \in \mathbb{W}_N^{\mathrm{A}}$ does not hit the boundary of $\mathbb{W}_N^{\mathrm{A}}$ up to time $t > 0$ is given by $\mathcal{N}^{\mathrm{A}}(t,\mathbf{x}) = \int_{\mathbb{W}_N^{\mathrm{A}}} d\mathbf{y}\, f^{\mathrm{A}}(t,\mathbf{y}|\mathbf{x})$. In the previous papers [31, 32], we gave the asymptotic form

$$f^{\mathrm{A}}(t,\mathbf{y}|\mathbf{x}) = \frac{t^{-N(N+1)/4}}{C[\mathrm{A}]} h^{\mathrm{A}}\left(\frac{\mathbf{x}}{\sqrt{t}}\right) h^{\mathrm{A}}(\mathbf{y}) \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \times \left(1+\mathcal{O}\left(\frac{|\mathbf{x}|}{\sqrt{t}}\right)\right) \tag{34}$$

in $|\mathbf{x}|/\sqrt{t} \to 0$ and showed that $\mathcal{N}^{\rm A}(t,\mathbf{x}) = (C[{\rm A}']/C[{\rm A}]) h^{\rm A}(\mathbf{x}/\sqrt{t}) \times (1+\mathcal{O}(|\mathbf{x}|/\sqrt{t}))$ as $|\mathbf{x}|/\sqrt{t} \to 0$. This estimate gives that for $\mathbf{x} \in \mathbb{W}_N^{\rm A}$ the noncolliding probability decays in the power-law as $t \to \infty$ [15, 21, 37]; $\mathcal{N}^{\rm A}(t,\mathbf{x}) \sim t^{-\psi[{\rm A}]}$ with the exponent $\psi[{\rm A}] = N(N-1)/4$. (Note that (34) is derived readily by using (A.1) in Appendix A.) For a given $T>0$, we defined

$$g_T^{\rm A}(s,\mathbf{x};t,\mathbf{y}) = \frac{f^{\rm A}(t-s,\mathbf{y}|\mathbf{x})\mathcal{N}^{\rm A}(T-t,\mathbf{y})}{\mathcal{N}^{\rm A}(T-s,\mathbf{x})}$$

for $0 \le s < t \le T$, $\mathbf{x},\mathbf{y} \in \mathbb{W}_N^{\rm A}$. Using (34) we showed that as $|\mathbf{x}| \to 0$ it converges to $g_T^{\rm A}(0,\mathbf{0};t,\mathbf{y}) = (T^{\psi[{\rm A}]} t^{-d[{\rm A}]/2}/C[{\rm A}'])e^{-|\mathbf{y}|^2/2t} h^{\rm A}(\mathbf{y})\mathcal{N}^{\rm A}(T-t,\mathbf{y})$. This function $g_T^{\rm A}(s,\mathbf{x};t,\mathbf{y})$ can be regarded as the transition probability density from the state $\mathbf{x} \in \mathbb{W}_N^{\rm A}$ at time s to the state $\mathbf{y} \in \mathbb{W}_N^{\rm A}$ at time $t(>s)$ conditioned to stay inside $\mathbb{W}_N^{\rm A}$ up to time T and defines a temporally inhomogeneous diffusion process, which we denoted by $\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_N(t))$, $t \in [0,T]$ in Sec.I. This represents the N-particle system of Brownian motions conditioned not to collide with each other in a finite time-interval $(0,T]$. The process $\mathbf{X}(t), t \in [0,T]$, starting from $\mathbf{X}(0) = \mathbf{0}$ is illustrated by Figure 1, whose spatial-temporal path-configuration is said to be in *star topology* in the theory of directed polymer networks [14]. As mentioned in Sec.I, this process exhibits a transition of the eigenvalue statistics from GUE to GOE [31, 32].

In the present section, we consider the temporally inhomogeneous diffusion process associated with $\mathbf{Y}^{(\nu)}(t)$ studied in the previous section. We consider the N-particle system of generalized meanders (5) conditioned that they never collide with each other for a time interval $[0,T]$. The transition probability density is given by

$$g_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{f_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y})\mathcal{N}_T^{(\nu,\kappa)}(t,\mathbf{y})}{\mathcal{N}_T^{(\nu,\kappa)}(s,\mathbf{x})} \tag{35}$$

for $0 \le s < t \le T, \mathbf{x},\mathbf{y} \in \mathbb{W}_N^{\rm C}$, where $f_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \det_{1\le i,j\le N}[G_T^{(\nu,\kappa)}(s,x_i;t,y_j)]$ with (5) and $\mathcal{N}_T^{(\nu,\kappa)}(t,\mathbf{x}) = \int_{\mathbb{W}_N^{\rm C}} d\mathbf{y} f_T^{(\nu,\kappa)}(t,\mathbf{x};T,\mathbf{y})$. Note that $f_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = f^{(\nu)}(t-s,\mathbf{y}|\mathbf{x})h_T^{(\nu,\kappa)}(t,\mathbf{y})/h_T^{(\nu,\kappa)}(s,\mathbf{x})$, where $h_T^{(\nu,\kappa)}(t,\mathbf{x}) = \prod_{i=1}^N h_T^{(\nu,\kappa)}(t,x_i)$. Since $\lim_{t\to 0} G^{(\nu)}(t,z|w) = \delta(z-w)\mathbf{1}(z \ge 0)$, $h_T^{(\nu,\kappa)}(T,\mathbf{x}) = \prod_{j=1}^N x_j^{-\kappa}$ for $\mathbf{x} \in \mathbb{W}_N^{\rm C}$, and then (35) can be written as

$$g_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\widetilde{\mathcal{N}}^{(\nu,\kappa)}(T-s,\mathbf{x})} f^{(\nu)}(t-s,\mathbf{y}|\mathbf{x})\widetilde{\mathcal{N}}^{(\nu,\kappa)}(T-t,\mathbf{y}) \tag{36}$$

with

$$\widetilde{\mathcal{N}}^{(\nu,\kappa)}(t,\mathbf{x}) = \int_{\mathbb{W}_N^{\rm C}} d\mathbf{y}\ f^{(\nu)}(t,\mathbf{y}|\mathbf{x}) \prod_{i=1}^N y_i^{-\kappa}. \tag{37}$$

Lemma 5 *Assume that $\nu > -1$ and $\kappa \in [0, 2(\nu+1))$. Let $\mathbf{x},\mathbf{y} \in \mathbb{W}_N^{\rm C}$.*

(i) *For $0 \le s < t \le T$, $\displaystyle\lim_{T\to\infty} g_T^{(\nu,\kappa)}(s,\mathbf{x};t,\mathbf{y}) = p^{(\nu)}(s,\mathbf{x};t,\mathbf{y})$.*

(ii) *For $0 < t < T$,*

$$\begin{aligned} g_T^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}) &\equiv \lim_{|\mathbf{x}|\to 0} g_T^{(\nu,\kappa)}(0,\mathbf{x};t,\mathbf{y}) \\ &= \frac{T^{N(N+\kappa-1)/2} t^{-N(N+\nu)}}{C_{\nu,\kappa}} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h^{(2\nu+1)}(\mathbf{y})\widetilde{\mathcal{N}}^{(\nu,\kappa)}(T-t,\mathbf{y}). \end{aligned} \tag{38}$$

(iii) *For $T > 0$,* $\displaystyle\lim_{t\nearrow T} g_T^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}) = \frac{T^{-N(N+2\nu+1-\kappa)/2}}{C_{\nu,\kappa}} \exp\left\{-\frac{|\mathbf{y}|^2}{2T}\right\} h^{(2\nu+1-\kappa)}(\mathbf{y})$.

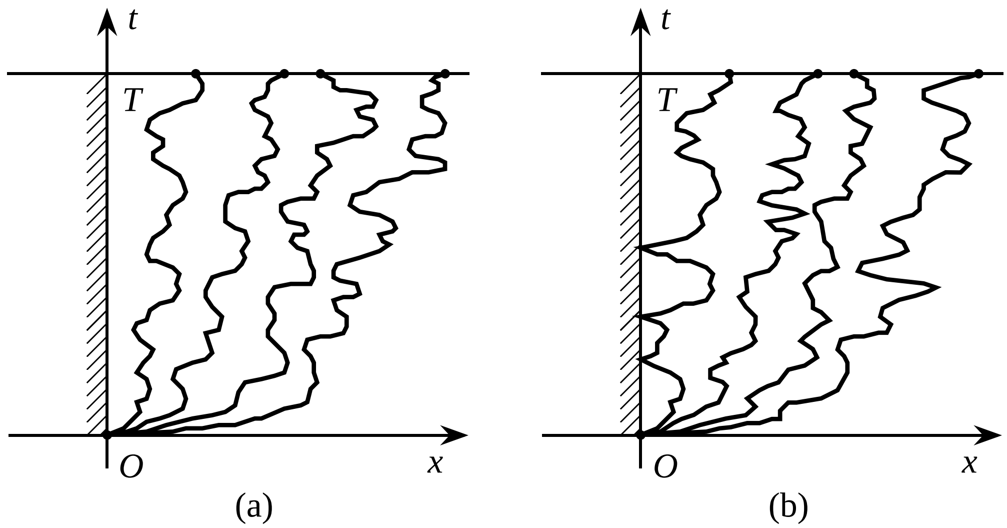

Figure 2: (a) Process $\mathbf{X}^{(1/2,1)}(t), t \in [0,T]$ with the initial state $\mathbf{0}$ showing star topology. (b) Process $\mathbf{X}^{(-1/2,0)}(t), t \in [0,T]$ with the initial state $\mathbf{0}$ showing star topology.

Proof. Using (33) for (37), we have the estimate of $\widetilde{\mathcal{N}}^{(\nu,\kappa)}(t,\mathbf{x})$ in $|\mathbf{x}|/\sqrt{t} \to 0$ as

$$\begin{aligned}
\widetilde{\mathcal{N}}^{(\nu,\kappa)}(t,\mathbf{x}) &= \frac{t^{-N(N+2\nu+1)/2}}{C_\nu} \prod_{1\le i<j\le N} \left\{ \left(\frac{x_j}{\sqrt{t}}\right)^2 - \left(\frac{x_i}{\sqrt{t}}\right)^2 \right\} \\
&\times \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{y} \prod_{1\le k\le \ell\le N} (y_\ell^2 - y_k^2) \prod_{m=1}^N y_m^{2\nu+1-\kappa} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} \times \left(1 + \mathcal{O}\left(\frac{|\mathbf{x}|}{\sqrt{t}}\right)\right) \\
&= \frac{t^{-N\kappa/2} C_{\nu,\kappa}}{C_\nu} \prod_{1\le i<j\le N} \left\{ \left(\frac{x_j}{\sqrt{t}}\right)^2 - \left(\frac{x_i}{\sqrt{t}}\right)^2 \right\} \times \left(1 + \mathcal{O}\left(\frac{|\mathbf{x}|}{\sqrt{t}}\right)\right), \qquad (39)
\end{aligned}$$

where we have used a version of Selberg's integral formula [50, 38]

$$\int_{\mathbb{R}^N} d\mathbf{u} \prod_{1\le i<j\le N} |u_j^2 - u_i^2|^{2\gamma} \prod_{k=1}^N |u_k|^{2\alpha-1} e^{-|\mathbf{u}|^2/2} = 2^{\alpha N + \gamma N(N-1)} \prod_{i=1}^N \frac{\Gamma(1+i\gamma)\Gamma(\alpha+\gamma(i-1))}{\Gamma(1+\gamma)}$$

by setting $\alpha = \nu + 1 - \kappa/2$ and $\gamma = 1/2$ (see Equation (17.6.6) in [40]). By (33) and (39), (i) and (ii) are obtained. Since $\lim_{t\to 0} G^{(\nu)}(t, y|x) = \delta(y-x)\mathbf{1}(y \ge 0)$, we have $\lim_{t\to 0} f^{(\nu)}(t,\mathbf{y}|\mathbf{x}) = \prod_{i=1}^N \delta(y_i - x_i)$ for $\mathbf{x},\mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}$. Then $\lim_{t\to 0} \widetilde{\mathcal{N}}^{(\nu,\kappa)}(t,\mathbf{x}) = \prod_{i=1}^N x_i^{-\kappa} \mathbf{1}(\mathbf{x} \in \mathbb{W}_N^{\mathrm{C}})$ and (iii) is obtained. ∎

Now we define the process $\mathbf{X}^{(\nu,\kappa)}(t) = (X_1^{(\nu,\kappa)}(t), X_2^{(\nu,\kappa)}(t), \cdots, X_N^{(\nu,\kappa)}(t)), t \in [0,T]$, as the temporally inhomogeneous diffusion process, whose transition probability density is given by (35) for $0 \le s < t \le T, \mathbf{x},\mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}$ and (38) for $0 < t \le T, \mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}$. This process solves the stochastic differential equations

$$dX_i^{(\nu,\kappa)}(t) = dB_i(t) + \left[\frac{2\nu+1}{2X_i^{(\nu,\kappa)}(t)} + b_i^{(\nu,\kappa)}(T-t, \mathbf{X}^{(\nu,\kappa)}(t))\right] dt, \quad t \in [0,T], 1 \le i \le N,$$

where $b_i^{(\nu,\kappa)}(t,\mathbf{x}) = (\partial/\partial x_i) \ln \widetilde{\mathcal{N}}^{(\nu,\kappa)}(t,\mathbf{x})$, $1 \le i \le N$.

Here we consider the special cases $(\nu,\kappa) = (1/2, 1)$ and $(\nu,\kappa) = (-1/2, 0)$. By the definitions (29) and (37), $\widetilde{\mathcal{N}}^{(1/2,1)}(t,\mathbf{x}) = \mathcal{N}^{\mathrm{C}}(t,x)/\prod_{i=1}^N x_i$ and $\widetilde{\mathcal{N}}^{(-1/2,0)}(t,\mathbf{x}) = \mathcal{N}^{\mathrm{D}}(t,x)$, and then (36) gives

$$g_T^{(1/2,1)}(s,\mathbf{x};t,\mathbf{y}) = \frac{1}{\mathcal{N}^{\mathrm{C}}(T-s,\mathbf{x})} f^{\mathrm{C}}(t-s,\mathbf{y}|\mathbf{x}) \mathcal{N}^{\mathrm{C}}(T-t,\mathbf{y}),$$

$$g_T^{(-1/2,0)}(s,\mathbf{x};t,\mathbf{y}) \quad = \quad \frac{1}{\mathcal{N}^{\rm D}(T-s,\mathbf{x})} f^{\rm D}(t-s,\mathbf{y}|\mathbf{x}) \mathcal{N}^{\rm D}(T-t,\mathbf{y}),$$

for $0 \leq s < t \leq T, \mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\rm C}$. That is, we can interpret the process $\mathbf{X}^{(1/2,1)}(t)$ as the N-particle system of Brownian motions conditioned never to collide with each other nor with the wall at the origin in one-dimension during the time-interval $[0,T]$, and the process $\mathbf{X}^{(-1/2,0)}(t)$ as the N-particle system of reflecting Brownian motions conditioned never to collide with each other during the time-interval $[0,T]$, respectively. The asymptotic forms $\mathcal{N}^\sharp(t,\mathbf{x}) = (C[\sharp']/C[\sharp]) h^\sharp(\mathbf{x}/\sqrt{t}) \times (1+\mathcal{O}(|\mathbf{x}|/\sqrt{t}))$ in $|\mathbf{x}|/\sqrt{t} \to 0$ for $\sharp$ =C and D are obtained by (39), and thus we can see the power-laws of the noncolliding probabilities, $\mathcal{N}^\sharp(t,\mathbf{x}) \sim t^{-\psi[\sharp]}$ as $t \to \infty$ for $\mathbf{x} \in \mathbb{W}_N^\sharp, \sharp =$ C and D with the exponents $\psi[{\rm C}] = N^2/2, \psi[{\rm D}] = N(N-1)/2$. As a corollary of Lemma 5, we have the following.

Corollary 6 *(i)* *For $0 < t < T$, if $\mathbf{x} \in \mathbb{W}_N^{\rm C}$,*

$$\begin{aligned} g_T^{(1/2,1)}(0,\mathbf{0};t,\mathbf{x}) &= \frac{T^{\psi[{\rm C}]} t^{-d[{\rm C}]/2}}{C[{\rm C}']} \exp\left\{-\frac{|\mathbf{x}|^2}{2t}\right\} h^{\rm C}(\mathbf{x}) \mathcal{N}^{\rm C}(T-t,\mathbf{x}), \\ g_T^{(-1/2,0)}(0,\mathbf{0};t,\mathbf{x}) &= \frac{T^{\psi[{\rm D}]} t^{-d[{\rm D}]/2}}{C[{\rm D}']} \exp\left\{-\frac{|\mathbf{x}|^2}{2t}\right\} h^{\rm D}(\mathbf{x}) \mathcal{N}^{\rm D}(T-t,\mathbf{x}). \end{aligned}$$

(ii) *For $T > 0$, if $\mathbf{x} \in \mathbb{W}_N^{\rm C}$,*

$$\lim_{t \nearrow T} g_T^{(1/2,1)}(0,\mathbf{0};t,\mathbf{x}) = q^{{\rm C}'}(\mathbf{x};T), \quad \lim_{t \nearrow T} g_T^{(-1/2,0)}(0,\mathbf{0};t,\mathbf{x}) = q^{{\rm D}'}(\mathbf{x};T).$$

Figure 2 illustrates the processes $\mathbf{X}^{(1/2,1)}(t)$ and $\mathbf{X}^{(-1/2,0)}(t)$ both starting from $\mathbf{0}$. The path-configurations are in star topology. In the former any particle can not collide with the wall at the origin, while in the latter the leftmost particle is reflected at the wall. Another corollary of Lemma 5 is the following.

Corollary 7 *If $\nu \in \mathbb{N}, \mathbf{x} \in \mathbb{W}_N^{\rm C}$,* $\displaystyle\lim_{t \nearrow T} g_T^{(\nu,\nu+1)}(0,\mathbf{0};t,\mathbf{x}) = q_\nu^{\rm chGOE}(\mathbf{x};T)$.

The combination of Lemma 5 (i) with (31) and (32) of Lemma 4, Corollaries 6 and 7 implies that $\mathbf{X}^{(1/2,1)}(t)$, $\mathbf{X}^{(-1/2,0)}(t)$ and $\mathbf{X}^{(\nu,\nu+1)}(t)$ with $\nu \in \mathbb{N}$, all starting from $\mathbf{0}$, exhibit the transitions from the eigenvalue statistics of the class C to the class CI, from the class D to the class associated with $q^{{\rm D}'}$ studied in Sec.III.D, and from chGUE to chGOE, respectively, as time t goes on from 0 to T. (See Theorem 9 below.)

At the end of this subsection, we discuss the relation between the temporally homogeneous diffusion process $\mathbf{Y}^{(\nu)}(t)$ and the temporally inhomogeneous diffusion process $\mathbf{X}^{(\nu,\kappa)}(t)$ for $t \in [0,T]$. For a time sequence $t_0 \equiv 0 < t_1 < \cdots < t_{\ell-1} < t_\ell \equiv T < \infty$ with $\ell \in \{1,2,\cdots\}$, we consider the multi-time probabilities with the initial state $\mathbf{Y}^{(\nu)}(0) = \mathbf{X}^{(\nu,\kappa)}(0) = \mathbf{x}^{(0)}$

$$P^{\mathbf{x}^{(0)}}\left(\mathbf{Y}^{(\nu)}(t_1) \in d\mathbf{x}^{(1)}, \cdots, \mathbf{Y}^{(\nu)}(t_\ell) \in d\mathbf{x}^{(\ell)}\right) = \prod_{i=1}^{\ell} p^{(\nu)}(t_{i-1},\mathbf{x}^{(i-1)};t_i,\mathbf{x}^{(i)}) d\mathbf{x}^{(i)},$$

and

$$P^{\mathbf{x}^{(0)}}\left(\mathbf{X}^{(\nu,\kappa)}(t_1) \in d\mathbf{x}^{(1)}, \cdots, \mathbf{X}^{(\nu,\kappa)}(t_\ell) \in d\mathbf{x}^{(\ell)}\right) = \prod_{i=1}^{\ell} g_T^{(\nu,\kappa)}(t_{i-1},\mathbf{x}^{(i-1)};t_i,\mathbf{x}^{(i)}) d\mathbf{x}^{(i)},$$

where we have used the Markov property of the processes. Assume that $\mathbf{x}^{(0)} = \mathbf{0}$ and $\mathbf{x}^{(i)} \in \mathbb{W}_N^{\rm C}, 1 \leq i \leq \ell$. We use the formulae (26) and (36) and apply Lemmas 4 and 5. Then we have the equality

$$\prod_{i=1}^{\ell} g_T^{(\nu,\kappa)}(t_{i-1},\mathbf{x}^{(i-1)};t_i,\mathbf{x}^{(i)}) = T^{N(N+\kappa-1)/2} \frac{C_\nu}{C_{\nu,\kappa}} \prod_{i=1}^{\ell} p^{(\nu)}(t_{i-1},\mathbf{x}^{(i-1)};t_i,\mathbf{x}^{(i)}) \frac{1}{h^{(\kappa)}(\mathbf{x}^{(\ell)})}.$$

Since this equality holds for arbitrary time sequence $t_0 = 0 < t_1 < \cdots < t_{\ell-1} < t_\ell = T < \infty$ with $\ell \in \{1,2,\cdots\}$, we can conclude the following.

Proposition 8 *Assume that* $\nu > -1, \kappa \in [0, 2(\nu+1))$. *If* $\mathbf{X}^{(\nu,\kappa)}(0) = \mathbf{Y}^{(\nu)}(0) = \mathbf{0}$, *then the distribution of the process* $\mathbf{X}^{(\nu,\kappa)}(t)$ *is absolutely continuous with that of the process* $\mathbf{Y}^{(\nu)}(t)$ *for* $t \in [0,T]$ *and the Radon-Nikodým density is given by*

$$\frac{P(\mathbf{X}^{(\nu,\kappa)}(\cdot) \in d\mathbf{w})}{P(\mathbf{Y}^{(\nu)}(\cdot) \in d\mathbf{w})} = \frac{C_\nu T^{N(N+\kappa-1)/2}}{C_{\nu,\kappa} h^{(\kappa)}(\mathbf{w}(T))}.$$

When $N = 1$ and $(\nu, \kappa) = (1/2, 1)$, this proposition gives the Imhof relation between the Brownian meander and the three-dimensional Bessel process [25]. The relation stated by (4) [31, 32] and the above proposition are regarded as the multivariate generalizations of the Imhof relation.

B Brownian bridges and temporally inhomogeneous matrix-valued processes

Assume that $\nu \in \mathbb{N}, 0 < T < \infty$. Let $B_{ij}^\rho(t), \widetilde{B}_{ij}^\rho(t), 1 \le i \le N+\nu, 1 \le j \le N, 0 \le \rho \le 3$ be independent one-dimensional standard Brownian motions. For a given matrix $m = (m_{ij} + \sqrt{-1}\widetilde{m}_{ij})_{1\le i\le N+\nu, 1\le j\le N}$ with $m_{ij}, \widetilde{m}_{ij} \in \mathbb{R}$, let $(\beta_T^\rho)_{ij}(t : m_{ij}), (\widetilde{\beta}_T^\rho)_{ij}(t : \widetilde{m}_{ij}), 1 \le i \le N+\nu, 1 \le j \le N, 0 \le \rho \le 3$ be the diffusion processes, which are the solutions of the following stochastic differential equations:

$$\begin{aligned}
(\beta_T^\rho)_{ij}(t : m_{ij}) &= B_{ij}^\rho(t) - \int_0^t \frac{(\beta_T^\rho)_{ij}(s : m_{ij}) - m_{ij}}{T-s} ds, \\
(\widetilde{\beta}_T^\rho)_{ij}(t : \widetilde{m}_{ij}) &= \widetilde{B}_{ij}^\rho(t) - \int_0^t \frac{(\widetilde{\beta}_T^\rho)_{ij}(s : \widetilde{m}_{ij}) - \widetilde{m}_{ij}}{T-s} ds, \quad t \in [0,T].
\end{aligned} \tag{40}$$

The processes $(\beta_T^\rho)_{ij}(t : m_{ij})$ and $(\widetilde{\beta}_T^\rho)_{ij}(t : \widetilde{m}_{ij})$ are one-dimensional Brownian bridges of duration T both starting from 0 and ending at m_{ij} and $\widetilde{m}_{ij}$, respectively [55]. Next for $z^\rho = (z_{ij}^\rho)_{1\le i,j\le N} \in \mathcal{S}(N)$ and $\widetilde{z}^\rho = (\widetilde{z}_{ij}^\rho)_{1\le i,j\le N} \in \mathcal{A}(N)$, $0 \le \rho \le 3$, we set

$$(s_T^\rho)_{ij}(t : z_{ij}^\rho) = \begin{cases} \dfrac{1}{\sqrt{2}}(\beta_T^\rho)_{ij}(t : \sqrt{2} z_{ij}^\rho), & \text{if } i < j, \\[2ex] (\beta_T^\rho)_{ii}(t : z_{ii}^\rho), & \text{if } i = j, \end{cases}$$

and

$$(a_T^\rho)_{ij}(t : \widetilde{z}_{ij}^\rho) = \begin{cases} \dfrac{1}{\sqrt{2}}(\widetilde{\beta}_T^\rho)_{ij}(t : \sqrt{2} \widetilde{z}_{ij}^\rho), & \text{if } i < j, \\[2ex] 0, & \text{if } i = j, \end{cases} \tag{41}$$

with $(s_T^\rho)_{ij}(t : z_{ij}^\rho) = (s_T^\rho)_{ji}(t : z_{ji}^\rho)$ and $(a_T^\rho)_{ij}(t : \widetilde{z}_{ij}^\rho) = -(a_T^\rho)_{ji}(t : \widetilde{z}_{ji}^\rho)$ for $i > j$, where $1 \le i, j \le N, 0 \le \rho \le 3$ and $t \in [0,T]$. We define the matrix-valued processes $s_T^\rho(t : z^\rho) = ((s_T^\rho)_{ij}(t : z_{ij}^\rho))_{1\le i,j\le N} \in \mathcal{S}(N)$ and $a_T^\rho(t : \widetilde{z}^\rho) = ((a_T^\rho)_{ij}(t : \widetilde{z}_{ij}^\rho))_{1\le i,j\le N} \in \mathcal{A}(N)$.

In an earlier paper [33], we considered the $N \times N$ hermitian matrix-valued process $\Xi_T(t) = s^0(t) + \sqrt{-1} a_T^0(t : O), t \in [0,T]$, where O denotes the $N \times N$ zero matrix and $s^0(t)$ was defined below (17). This process is the temporally inhomogeneous matrix-valued process realized as an interpolation in duration T of the first and second processes given in Sec.II.B. Using the invariance in distribution of the process $\Xi_T(t)$ under unitary transformations and our generalized version of the Imhof relation (4), we proved the equivalence in distribution of its eigenvalue process and $\mathbf{X}(t)$ with $\mathbf{X}(0) = \mathbf{0}$. As a corollary of this equivalence, we derived the formula for any $\sigma \in \mathbb{R}$,

$$\int_{\mathrm{U}(N)} dU \, \exp\left\{ -\frac{1}{2\sigma^2} \mathrm{Tr}(\Lambda_{\mathbf{x}} - U^\dagger \Lambda_{\mathbf{y}} U)^2 \right\} = \frac{C[\mathrm{A}]\sigma^{d[\mathrm{A}]}}{h^{\mathrm{A}}(\mathbf{x}) h^{\mathrm{A}}(\mathbf{y})} \det_{1\le i,j\le N} \left[G^{\mathrm{A}}(t, y_j | x_i) \right], \tag{42}$$

where dU denotes the Haar measure of $\mathrm{U}(N)$ normalized as $\int_{\mathrm{U(N)}} dU = 1$, $\Lambda_{\mathbf{x}} = \mathrm{diag}\{x_1, \cdots, x_N\}$ and $\Lambda_{\mathbf{y}} = \mathrm{diag}\{y_1, \cdots, y_N\}$ with $\mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\mathrm{A}}$. This is a stochastic-calculus derivation of the Harish-Chandra (Itzykson-Zuber) integral formula [22, 26]. In this subsection, we give extensions of this argument.

As an interpolation of the Laguerre process (7) and the Wishart process (9), we define the matrix-valued process

$$\Xi_T^{\rm LW}(t)=M_T(t)^\dagger M_T(t),\quad t\in[0,T],$$

where $M_T(t)=(B_{ij}^0(t)+\sqrt{-1}(\widetilde{\beta}_T^0)_{ij}(t:O))_{1\le i\le N+\nu,1\le j\le N}\in\mathcal{M}(N+\nu,N;\mathbb{C}),t\in[0,T]$, where O denotes the $(N+\nu)\times N$ zero matrix. Similarly, the interpolations between the processes (21) and (22) are defined by

$$\Xi_T^{\rm C}(t)=\sqrt{-1}a_T^0(t:O)\otimes\sigma_0+s^1(t)\otimes\sigma_1+s_T^2(t:O)\otimes\sigma_2+s^3(t)\otimes\sigma_3,$$
$$\Xi_T^{\rm D}(t)=\sqrt{-1}a_T^0(t:O)\otimes\sigma_0+\sqrt{-1}a_T^1(t:O)\otimes\sigma_1+\sqrt{-1}a^2(t)\otimes\sigma_2+s^3(t)\otimes\sigma_3,$$

in which O denotes the $N\times N$ zero matrix. Let $\boldsymbol{\kappa}^{\rm LW}(t)=(\kappa_1^{\rm LW}(t),\cdots,\kappa_N^{\rm LW}(t)),t\in[0,T]$ be the square roots of the eigenvalues of $\Xi_T^{\rm LW}(t)$ with $0\le\kappa_1^{\rm LW}(t)\le\cdots\le\kappa_N^{\rm LW}(t)$ and $\boldsymbol{\lambda}^\sharp(t)=(\lambda_1^\sharp(t),\lambda_2^\sharp(t),\cdots,\lambda_N^\sharp(t))$ be the nonnegative eigenvalues of $\Xi_T^\sharp(t)$ with $0\le\lambda_1^\sharp(t)\le\cdots\le\lambda_N^\sharp(t)$ for $\sharp=$ C and D. We prove the following equivalence in distribution among the temporally inhomogeneous diffusion processes.

Theorem 9 *(i) If $\nu\in\mathbb{N}$ and $\mathbf{X}^{(\nu,\nu+1)}(0)=\mathbf{0}$, then $\boldsymbol{\kappa}^{\rm LW}(t)=\mathbf{X}^{(\nu,\nu+1)}(t)$, $t\in[0,T]$ in distribution.*

(ii) If $\mathbf{X}^{(1/2,1)}(0)=\mathbf{X}^{(-1/2,0)}(0)=\mathbf{0}$, then $\boldsymbol{\lambda}^{\rm C}(t)=\mathbf{X}^{(1/2,1)}(t)$ and $\boldsymbol{\lambda}^{\rm D}(t)=\mathbf{X}^{(-1/2,0)}(t)$, $t\in[0,T]$ in distribution.

Proof. (i) For a given matrix $m=(m_{ij}+\sqrt{-1}\widetilde{m}_{ij})_{1\le i\le N+\nu,1\le j\le N}$, $m_{ij},\widetilde{m}_{ij}\in\mathbb{R}$, we consider $\mathcal{M}(N+\nu,N;\mathbb{C})$-valued process $M_T(t:m)=((\beta_T^0)_{ij}(t:m_{ij})+\sqrt{-1}(\widetilde{\beta}_T^0)_{ij}(t:\widetilde{m}_{ij}))_{1\le i\le N+\nu,1\le j\le N},t\in[0,T]$. From the equations (40), we have the equation

$$M_T(t:m)=M(t)-\int_0^t\frac{M_T(s:m)-m}{T-s}ds,\quad t\in[0,T],\tag{43}$$

where $M(t)=(B_{ij}^0(t)+\sqrt{-1}\widetilde{B}_{ij}^0(t))_{1\le i\le N+\nu,1\le j\le N}$. Let m_U and m_O be random matrices with distribution $\mu_\nu^{\rm chGUE}(\,\cdot\,;T)$ and $\mu_\nu^{\rm chGOE}(\,\cdot\,;T)$, respectively. Since $(\beta_T^0)_{ij}(t:\zeta)$ and $(\widetilde{\beta}_T^0)_{ij}(t:\zeta),t\in[0,T]$ are Brownian motions when ζ is a Gaussian random variable with variance T independent of $B_{ij}^0(t)$ and $\widetilde{B}_{ij}^0(t)$, if m_U and m_O are independent of $M(t),t\in[0,T]$,

$$M_T(t:m_U)=M(t),\quad M_T(t:m_O)=M_T(t),\quad t\in[0,T]\tag{44}$$

in distribution. Moreover, since the distribution of the process $M(t)$ is invariant under any transformation $M(t)\to U^\dagger M(t)V$, $U\in{\rm U}(N+\nu),V\in{\rm U}(N)$, the following lemma is obtained by the equation (43).

Lemma 10 *For any $U\in{\rm U}(N+\nu),V\in{\rm U}(N)$, $U^\dagger M_T(t:m)V=M_T(t:U^\dagger mV)$, $t\in[0,T]$ in distribution*

By this lemma, if m and m' in $\mathcal{M}(N+\nu,N;\mathbb{C})$ have the same radial coordinates, the processes of radial coordinates of $M_T(t:m)$ and $M_T(t:m'),t\in[0,T]$, are identical in distribution. Let $\Xi_T^{\rm LW}(t:m)=M_T^\dagger(t:m)M_T(t:m)$. Then the above gives the identification in distribution of the processes of square roots of eigenvalues of $\Xi_T^{\rm LW}(t:m)$ and $\Xi_T^{\rm LW}(t:m'),t\in[0,T]$. Now we denote by $P_T^{\boldsymbol{\kappa}}(\cdot)$ the probability distribution of the process of square roots of eigenvalues of $\Xi_T^{\rm LW}(t:m)$ conditioned that the square roots of eigenvalues of m is $\boldsymbol{\kappa}=(\kappa_1,\cdots,\kappa_N)$ with the condition (11). We also denote by $P(\cdot)$ and $P_T(\cdot)$ the distributions of the processes of square roots of eigenvalues of $\Xi(t)=M(t)^\dagger M(t)$ and $\Xi_T(t)=M_T^\dagger(t)M_T(t),t\in[0,T]$, respectively. The equalities (44) give

$$P(\cdot)=\int_{\mathbb{W}_N^{\rm C}}d\boldsymbol{\kappa}\,P_T^{\boldsymbol{\kappa}}(\cdot)q_\nu^{\rm chGUE}(\boldsymbol{\kappa};T),\quad P_T(\cdot)=\int_{\mathbb{W}_N^{\rm C}}d\boldsymbol{\kappa}\,P_T^{\boldsymbol{\kappa}}(\cdot)q_\nu^{\rm chGOE}(\boldsymbol{\kappa};T).$$

Then $P_T(\cdot)$ and $P(\cdot)$ satisfy the same relation as the generalized Imhof relation between $\mathbf{X}^{(\nu,\nu+1)}(t)$ and $\mathbf{Y}^{(\nu)}(t)$ obtained from Proposition 8 by setting $\nu\in\mathbb{N},\kappa=\nu+1$. Since $P(\cdot)$ is equal to the distribution of

the temporally homogeneous diffusion process $\mathbf{Y}^{(\nu)}(t)$ (see (31) of Lemma 4), we can conclude that $P_T(\cdot)$ is identical to the distribution of the process $\mathbf{X}^{(\nu,\nu+1)}(t)$.

(ii) The second part can be proved by the same argument as the first part. For given $y^\rho, z^\rho \in \mathcal{S}(N)$, $\widetilde{y}^\rho$, $\widetilde{z}^\rho \in \mathcal{A}(N)$, $0 \le \rho \le 3$, put $Y = \sqrt{-1}\widetilde{y}^0 \otimes \sigma_0 + y^1 \otimes \sigma_1 + y^2 \otimes \sigma_2 + y^3 \otimes \sigma_3 \in \mathcal{H}_{2-}(2N)$ and $Z = \sqrt{-1}\widetilde{z}^0 \otimes \sigma_0 + \sqrt{-1}\widetilde{z}^1 \otimes \sigma_1 + \sqrt{-1}\widetilde{z}^2 \otimes \sigma_2 + z^3 \otimes \sigma_3 \in \mathcal{H}_{1-}(2N)$. For these Y and Z, we introduce the temporally inhomogeneous matrix-valued processes

$$\begin{aligned}\Xi_T^{\rm C}(t:Y) &= \sqrt{-1}a_T^0(t:\widetilde{y}^0) \otimes \sigma_0 + s_T^1(t:y^1) \otimes \sigma_1 + s_T^2(t:y^2) \otimes \sigma_2 + s_T^3(t:y^3) \otimes \sigma_3, \\ \Xi_T^{\rm D}(t:Z) &= \sqrt{-1}a_T^0(t:\widetilde{z}^0) \otimes \sigma_0 + \sqrt{-1}a_T^1(t:\widetilde{z}^1) \otimes \sigma_1 + \sqrt{-1}a_T^2(t:\widetilde{z}^2) \otimes \sigma_2 + s_T^3(t:z^3) \otimes \sigma_3.\end{aligned}$$

The key lemma 10 of the proof is replaced by the following.

Lemma 11 *For any $U \in {\rm U}_2(2N), V \in {\rm U}_1(2N)$, $U^\dagger \Xi_T^{\rm C}(t:Y)U = \Xi_T^{\rm C}(t:U^\dagger Y U)$, and $V^\dagger \Xi_T^{\rm D}(t:Z)V = \Xi_T^{\rm D}(t:V^\dagger Z V)$, $t \in [0,T]$ in distribution*

For $\sharp =$C and D we denote by $P_T^{\sharp,\boldsymbol{\omega}}(\cdot)$ the probability distributions of the processes of nonnegative eigenvalues of $\Xi_T^\sharp(t:Z)$ conditioned that the nonnegative eigenvalues of Z is $\boldsymbol{\omega} = (\omega_1, \cdots, \omega_N)$ with (18). We also denote by $P^\sharp(\cdot)$ and $P_T^\sharp(\cdot)$ the distributions of the processes of nonnegative eigenvalues of $\Xi^\sharp(t)$ and $\Xi_T^\sharp(t), t \in [0,T]$, respectively. Then we have the expressions,

$$P^\sharp(\cdot) = \int_{\mathbb{W}_N^{\rm C}} d\boldsymbol{\omega}\, P_T^{\sharp,\boldsymbol{\omega}}(\cdot) q^\sharp(\boldsymbol{\omega};T), \quad P_T^\sharp(\cdot) = \int_{\mathbb{W}_N^{\rm C}} d\boldsymbol{\omega}\, P_T^{\sharp,\boldsymbol{\omega}}(\cdot) q^{\sharp'}(\boldsymbol{\omega};T) \quad \text{for} \quad \sharp = \text{C, D}.$$

Comparing them with the $(\nu,\kappa) = (1/2,1)$ and $(\nu,\kappa) = (-1/2,0)$ cases of the generalized Imhof relations obtained from Proposition 8, we have the theorem. ∎

As a corollary of Theorem 9, the following integral formulae are derived as proved in Appendix B.

Corollary 12 *(i) Assume $\nu \in \mathbb{N}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\rm C}$. For any $\sigma \in \mathbb{R}$,*

$$\begin{aligned}&\int_{{\rm U}(N+\nu)\times{\rm U}(N)} d\mu(U,V)\, \exp\left\{-\frac{1}{2\sigma^2}{\rm Tr}(K_{\mathbf{x}} - U^\dagger K_{\mathbf{y}} V)^\dagger (K_{\mathbf{x}} - U^\dagger K_{\mathbf{y}} V)\right\} \\ &\qquad = \frac{C_\nu \sigma^{N(N+\nu-2)}}{h^{(\nu)}(\mathbf{x})h^{(\nu)}(\mathbf{y})} \det_{1\le i,j\le N}\left[e^{-(x_i^2+y_j^2)/2\sigma^2} I_\nu\left(\frac{x_i y_j}{\sigma^2}\right)\right],\end{aligned}$$

where

$$K_{\mathbf{x}} = \begin{pmatrix} \widehat{K}_{\mathbf{x}} \\ O \end{pmatrix}, \quad K_{\mathbf{y}} = \begin{pmatrix} \widehat{K}_{\mathbf{y}} \\ O \end{pmatrix},$$

with $\widehat{K}_{\mathbf{x}} = {\rm diag}\{x_1, x_2, \cdots, x_N\}, \widehat{K}_{\mathbf{y}} = {\rm diag}\{y_1, y_2, \cdots, y_N\}$ and $\nu \times N$ zero matrix O.

(ii) Let $\sharp =$ C, D. For $\mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\rm C}$, $\sigma \in \mathbb{R}$,

$$\int_{\widetilde{\rm U}(2N)} dU\, \exp\left\{-\frac{1}{4\sigma^2}{\rm Tr}(\Lambda_{\mathbf{x}} - U^\dagger \Lambda_{\mathbf{y}} U)^2\right\} = \frac{C[\sharp]\sigma^{d[\sharp]}}{h^\sharp(\mathbf{x})h^\sharp(\mathbf{y})} \det_{1\le i,j\le N}\left[G^\sharp(\sigma^2, y_j|x_i)\right],$$

where $\Lambda_{\mathbf{x}} = {\rm diag}\{x_1, x_2, \ldots, x_N\} \otimes \sigma_3$, $\Lambda_{\mathbf{y}} = {\rm diag}\{y_1, y_2, \ldots, y_N\} \otimes \sigma_3, \widetilde{\rm U}(2N) = {\rm U}_2(2N)$ for $\sharp =$ C and $\widetilde{\rm U}(2N) = {\rm U}_1(2N)$ for $\sharp =$ D.

They are extensions of the Harish-Chandra (Itzykson-Zuber) formula (42). The formula (i) is found in [27].

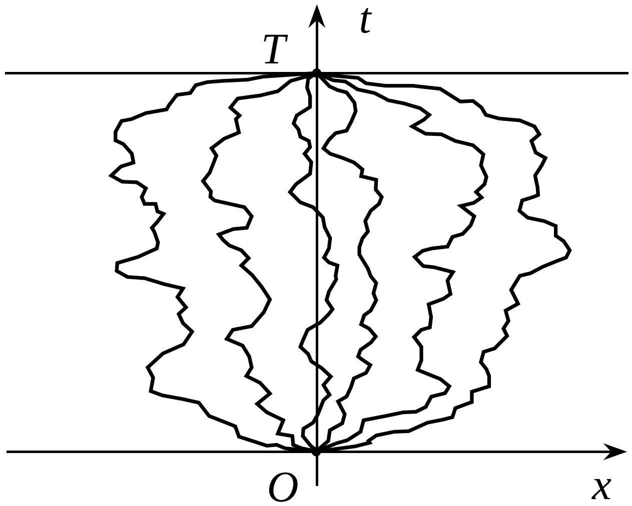

Figure 3: Process $\mathbf{X}^{\mathrm{A,w}}(t), t \in [0,T]$, showing watermelon topology.

C Watermelon topology

Consider the N-particle system of Brownian motions starting from $\mathbf{x} \in \mathbb{W}_N^{\mathrm{A}}$ at time $t=0$ and arriving at $\mathbf{z} \in \mathbb{W}_N^{\mathrm{A}}$ at time $T>0$, which do not collide with each other during the time interval $[0,T]$. We denote by $g^{\mathrm{A,w}}(0,\mathbf{x};t,\mathbf{y};T,\mathbf{z})$ the probability density of the state $\mathbf{y}$ at time $t \in [0,T]$. It is given by

$$g^{\mathrm{A,w}}(0,\mathbf{x};t,\mathbf{y};T,\mathbf{z}) = \frac{f^{\mathrm{A}}(t,\mathbf{y}|\mathbf{x})f^{\mathrm{A}}(T-t,\mathbf{z}|\mathbf{y})}{f^{\mathrm{A}}(T,\mathbf{z}|\mathbf{x})}, \quad \mathbf{y} \in \mathbb{W}_N^{\mathrm{A}}, t \in [0,T]. \tag{45}$$

By using (34), we can obtain the limit $g^{\mathrm{A,w}}(0,\mathbf{0};t,\mathbf{y};T,\mathbf{0}) = \lim_{|\mathbf{x}|\to 0, |\mathbf{z}|\to 0} g^{\mathrm{A,w}}(0,\mathbf{x};t,\mathbf{y};T,\mathbf{z})$. Let $\sigma_T(t) = \sqrt{t(1-t/T)}$.

Proposition 13 *For* $\mathbf{y} \in \mathbb{W}_N^{\mathrm{A}}$, $\quad g^{\mathrm{A,w}}(0,\mathbf{0};t,\mathbf{y};T,\mathbf{0}) = q^{\mathrm{GUE}}(\mathbf{y};\sigma_T(t)^2), t \in [0,T]$.

We denote by $\mathbf{X}^{\mathrm{A,w}}(t), t \in [0,T]$, the temporally inhomogeneous diffusion process, whose probability density is given by the above. Its path-configuration on the spatio-temporal plane is illustrated by Figure 3. Such a pattern is called *watermelon topology* in the polymer network theory [14].

For $\nu > -1$, similarly to (45) we put

$$g^{(\nu),\mathrm{w}}(0,\mathbf{x};t,\mathbf{y};T,\mathbf{z}) = \frac{f^{(\nu)}(t,\mathbf{y}|\mathbf{x})f^{(\nu)}(T-t,\mathbf{z}|\mathbf{y})}{f^{(\nu)}(T,\mathbf{z}|\mathbf{x})}$$

for $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{W}_N^{\mathrm{C}}, t \in [0,T]$. By (33) we have the following $\mathbf{x} \to \mathbf{0}$ limit.

Proposition 14 *For* $\nu > -1, \mathbf{x} \in \mathbb{W}_N^{\mathrm{C}}, t \in [0,T]$,

$$g^{(\nu),\mathrm{w}}(0,\mathbf{0};t,\mathbf{x};T,\mathbf{0}) = \frac{\sigma_T(t)^{-2N(N+\nu)}}{C_\nu} h^{((2\nu+1)/2)}(\mathbf{x})^2 \exp\left\{-\frac{|\mathbf{x}|^2}{2\sigma_T(t)^2}\right\}.$$

In particular, if $\nu \in \mathbb{N}$, $g^{(\nu),\mathrm{w}}(0,\mathbf{0};t,\mathbf{x};T,\mathbf{0}) = q_\nu^{\mathrm{chGUE}}(\mathbf{x},\sigma_T(t)^2)$, $g^{(1/2),\mathrm{w}}(0,\mathbf{0};t,\mathbf{x};T,\mathbf{0}) = q^{\mathrm{C}}(\mathbf{x},\sigma_T(t)^2)$, *and* $g^{(-1/2),\mathrm{w}}(0,\mathbf{0};t,\mathbf{x};T,\mathbf{0}) = q^{\mathrm{D}}(\mathbf{x},\sigma_T(t)^2)$.

We note that this expression may be formally obtained by taking $\kappa \to 2(\nu+1)$ limit of (38).

D Banana topology

For $\varepsilon > 0$, we consider a subspace of $\mathbb{W}_{2N}^{\mathrm{A}}$, $\mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon) = \{\mathbf{x} = (x_1, x_2, \cdots, x_{2N}) \in \mathbb{W}_{2N}^{\mathrm{A}} : x_{2i} = x_{2i-1} + \varepsilon, 1 \leq i \leq N\}$. For $\mathbf{x} \in \mathbb{W}_{2N}^{\mathrm{A}}$, we will use the notation $\mathbf{x}^{\mathrm{odd}} = (x_1, x_3, \cdots, x_{2N-1})$ and define $\mathcal{N}^{\mathrm{A,b}}(t, \mathbf{x}; \varepsilon) = \int_{\mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon)} d\mathbf{y}^{\mathrm{odd}} \, f^{\mathrm{A}}(t, \mathbf{y}|\mathbf{x})$. We consider the process, whose transition probability density is given by

$$g_T^{\mathrm{A,b}}(s, \mathbf{x}; t, \mathbf{y}; \varepsilon) = \frac{f^{\mathrm{A}}(t-s, \mathbf{y}|\mathbf{x}) \mathcal{N}^{\mathrm{A,b}}(T-t, \mathbf{y}; \varepsilon)}{\mathcal{N}^{\mathrm{A,b}}(T-s, \mathbf{x}; \varepsilon)}, \quad \mathbf{x}, \mathbf{y} \in \mathbb{W}_{2N}^{\mathrm{A}}, 0 \leq s < t \leq T.$$

This is the $2N$-particle system of noncolliding Brownian motions in $[0, T]$ conditioned that the final state at time $t = T$ is in $\mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon)$. Using (34),we have

$$g_T^{\mathrm{A,b}}(0, \mathbf{0}; t, \mathbf{y}; \varepsilon) \equiv \lim_{|\mathbf{x}| \to 0} g_T^{\mathrm{A,b}}(0, \mathbf{x}; t, \mathbf{y}; \varepsilon) = \left(\frac{t}{T} \right)^{-2N^2} \frac{h^{\mathrm{A}}(\mathbf{y}) e^{-|\mathbf{y}|^2/2t} \mathcal{N}^{\mathrm{A,b}}(T-t, \mathbf{y}; \varepsilon)}{\int_{\mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon)} d\mathbf{z}^{\mathrm{odd}} \, h^{\mathrm{A}}(\mathbf{z}) e^{-|\mathbf{z}|^2/2T}}$$

for $\mathbf{y} \in \mathbb{W}_{2N}^{\mathrm{A}}, t \in (0, T]$. Since $\lim_{t \to 0} f^{\mathrm{A}}(0, \mathbf{y}|\mathbf{x}) = \prod_{i=1}^N \delta(x_i - y_i)$, $\lim_{t \to 0} \mathcal{N}^{\mathrm{A,b}}(t, \mathbf{x}; \varepsilon) = \mathbf{1}(\mathbf{x} \in \mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon))$, and then for $\mathbf{y} \in \mathbb{W}_{2N}^{\mathrm{A}}$

$$\lim_{t \nearrow T} g_T^{\mathrm{A,b}}(0, \mathbf{0}; t, \mathbf{y}; \varepsilon) = \frac{h^{\mathrm{A}}(\mathbf{y}) e^{-|\mathbf{y}|^2/2T}}{\int_{\mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon)} d\mathbf{z}^{\mathrm{odd}} \, h^{\mathrm{A}}(\mathbf{z}) e^{-|\mathbf{z}|^2/2T}} \mathbf{1}(\mathbf{y} \in \mathbb{B}_{2N}^{\mathrm{A}}(\varepsilon)).$$

As implied in [41] we can take the limit, $g_T^{\mathrm{A,b}}(s, \mathbf{x}; t, \mathbf{y}) = \lim_{\varepsilon \to 0} g_T^{\mathrm{A,b}}(s, \mathbf{x}; t, \mathbf{y}; \varepsilon)$, in the above formulae to have

$$g_T^{\mathrm{A,b}}(s, \mathbf{x}; t, \mathbf{y}) = \frac{f^{\mathrm{A}}(t-s, \mathbf{y}|\mathbf{x}) \mathcal{N}^{\mathrm{A,b}}(T-t, \mathbf{y})}{\mathcal{N}^{\mathrm{A,b}}(T-s, \mathbf{x})}, \tag{46}$$

$$g_T^{\mathrm{A,b}}(0, \mathbf{0}; t, \mathbf{y}) = \left(\frac{T}{2} \right)^{N(2N+1)/2} \left(\frac{t}{2} \right)^{-2N^2} \frac{h^{\mathrm{A}}(\mathbf{y})}{C[\mathrm{A}'']} e^{-|\mathbf{y}|^2/2t} \mathcal{N}^{\mathrm{A,b}}(T-t, \mathbf{y}), \tag{47}$$

$$g_T^{\mathrm{A,b}}(0, \mathbf{0}; T, \mathbf{y}) = q^{\mathrm{GSE}} \left(\mathbf{y}^{\mathrm{odd}}; \frac{T}{2} \right) \mathbf{1}(\mathbf{y} \in \mathbb{B}_{2N}^{\mathrm{A}}), \tag{48}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{W}_{2N}^{\mathrm{A}}, 0 \leq s < t < T$, where $\mathcal{N}^{\mathrm{A,b}}(t, \mathbf{x}) = \int_{\mathbb{W}_N^{\mathrm{A}}} d\mathbf{y} \, f^{\mathrm{A,b}}(t, \mathbf{y}|\mathbf{x})$ with

$$f^{\mathrm{A,b}}(t, \mathbf{y}|\mathbf{x}) = \det_{1 \leq i \leq 2N, 1 \leq j \leq N} \left[G^{\mathrm{A}}(t, y_j|x_i) \quad \frac{x_i}{t} G^{\mathrm{A}}(t, y_j|x_i) \right]$$

for $\mathbf{x} \in \mathbb{W}_{2N}^{\mathrm{A}}$ and $\mathbf{y} \in \mathbb{W}_N^{\mathrm{A}}$, and $\mathbb{B}_{2N}^{\mathrm{A}} = \{\mathbf{x} = (x_1, x_2, \cdots, x_{2N}) : \mathbf{x}^{\mathrm{odd}} \in \mathbb{W}_N^{\mathrm{A}}, x_{2i} = x_{2i-1}, 1 \leq i \leq N\}$. We define the temporally inhomogeneous process $\mathbf{X}^{\mathrm{A,b}}(t), t \in [0, T]$ starting from $\mathbf{0}$ or the state in $\mathbb{W}_{2N}^{\mathrm{A}}$ and ending at the state in $\mathbb{B}_{2N}$ as the diffusion process, whose transition probability density is given by (46)-(48). The path-configuration of N particles in this version of noncolliding Brownian motions on the spatio-temporal plane is illustrated by Figure 4, which we would like to call "banana topology". Important point is that at the final time $t = T$ the particle positions are pairwise degenerated and distinct positions are identical in distribution with the Kramers doublets of eigenvalues of random matrices in GSE as claimed by (48).

Now we consider a $2N \times 2N$ hermitian matrix-valued temporally inhomogeneous process defined by

$$\begin{aligned} \Xi_T^{\mathrm{b}}(t) &= \left\{ s^0(t) + \sqrt{-1} a_T^0(t : O) \right\} \otimes \sigma_0 + \left\{ s_T^1(t : O) + \sqrt{-1} a^1(t) \right\} \otimes \sigma_1 \\ &+ \left\{ s_T^2(t : O) + \sqrt{-1} a^2(t) \right\} \otimes \sigma_2 + \left\{ s_T^3(t : O) + \sqrt{-1} a^3(t) \right\} \otimes \sigma_3, \end{aligned} \tag{49}$$

where the elements of the $N \times N$ matrices $\{s_T^\rho(t : z^\rho), (a_T^\rho : \widetilde{z}^\rho)\}_{\rho=1}^3$ are given by (41). By definition, $\Xi_T^{\mathrm{b}}(T)$ distributes with the probability density of GSE. Then the same argument as Theorem 9 may prove the following.

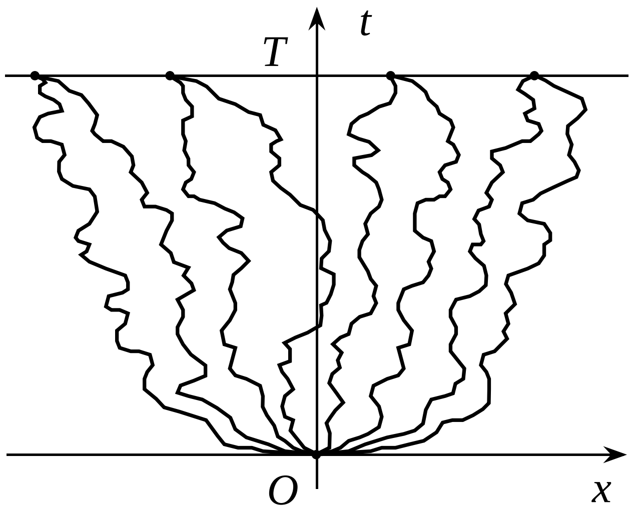

Figure 4: Process $\mathbf{X}^{\mathrm{A,b}}(t), t \in [0,T]$, showing banana topology.

Theorem 15 *Let $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_{2N}(t))$ be the eigenvalues of the process (49) with $\lambda_1(t) \le \lambda_2(t) \le \cdots \le \lambda_{2N}(t)$. If $\mathbf{X}^{\mathrm{A,b}}(0) = \mathbf{0}$, then $\boldsymbol{\lambda}(t) = \mathbf{X}^{\mathrm{A,b}}(t)$, $t \in [0,T]$ in distribution*

As a corollary of this theorem, we will have the following version of Harish-Chandra formula, which is found as Equation (3.46) in [41].

Corollary 16 *Let $\mathbf{x} = (x_1, x_2, \cdots, x_{2N}) \in \mathbb{W}^{\mathrm{A}}_{2N}$, $\mathbf{y} = (y_1, y_2, \cdots, y_N) \in \mathbb{W}^{\mathrm{A}}_N$. For any $\sigma \in \mathbb{R}$*

$$\int_{U(2N)} dU \, \exp\left\{ -\frac{1}{2\sigma^2} \mathrm{Tr}(\Lambda_{\mathbf{x}} - U^\dagger \Lambda_y U)^2 \right\} = \frac{C_{2N}[\mathrm{A}]\sigma^{(2N)^2}}{h^{\mathrm{A}}(\mathbf{x}) h^{\mathrm{A}}(\mathbf{y})^4} f^{\mathrm{A,b}}(\sigma^2, \mathbf{y}|\mathbf{x}),$$

where $\Lambda_{\mathbf{x}} = \mathrm{diag}\{x_1, x_2, \cdots, x_{2N}\}$, $\Lambda_{\mathbf{y}} = \mathrm{diag}\{y_1, y_2, \cdots, y_N\} \otimes \sigma_0$, and $C_{2N}[\mathrm{A}] = (2\pi)^N \prod_{i=1}^{2N} \Gamma(i)$.

It is easy to see by the same argument that the transition probability density given below defines the temporally inhomogeneous diffusion process $\mathbf{X}^{(\nu,\kappa),\mathrm{b}}(t), t \in [0,T], \nu > -1, \kappa \in [0, 2(\nu+1))$, associated with $\mathbf{X}^{(\nu,\kappa)}$, which shows the banana topology: Let

$$f^{(\nu),\mathrm{b}}(t, \mathbf{y}|\mathbf{x}) = \det_{1 \le i \le 2N, 1 \le j \le N} \begin{bmatrix} G^{(\nu)}(t, y_j|x_i) & G^{(\nu)}_y(t, y_j|x_i) \end{bmatrix}$$

for $\mathbf{x} \in \mathbb{W}^{\mathrm{C}}_{2N}, \mathbf{y} \in \mathbb{W}^{\mathrm{C}}_N$, where $G^{(\nu)}_y(t, y|x) = (\partial/\partial y) G^{(\nu)}(t, y|x)$, and let $\widetilde{\mathcal{N}}^{(\nu,\kappa),\mathrm{b}}(t, \mathbf{x}) = \int_{\mathbb{W}^{\mathrm{C}}_N} d\mathbf{y} \, f^{(\nu),\mathrm{b}}(t, \mathbf{y}|\mathbf{x}) \prod_{i=1}^N y_i^{-\kappa}$ for $\mathbf{x} \in \mathbb{W}^{\mathrm{C}}_{2N}$. Then

$$\begin{aligned}
g^{(\nu,\kappa),\mathrm{b}}_T(s, \mathbf{x}; t, \mathbf{y}) &= \frac{f^{(\nu)}(t-s, \mathbf{y}|\mathbf{x}) \widetilde{\mathcal{N}}^{(\nu,\kappa),\mathrm{b}}(T-t, \mathbf{y})}{\widetilde{\mathcal{N}}^{(\nu,\kappa),\mathrm{b}}(T-s, \mathbf{x})}, \\
g^{(\nu,\kappa),\mathrm{b}}_T(0, \mathbf{0}; t, \mathbf{y}) &= \frac{2^{N(4N+4\nu-1)} T^{2N^2} t^{-2N(2N+\nu)}}{\hat{C}_\nu} h^{(2\nu+1)}(\mathbf{y}) e^{-|\mathbf{y}|^2/2t} \widetilde{\mathcal{N}}^{(\nu,\kappa),\mathrm{b}}(T-t, \mathbf{y}), \\
g^{(\nu,\kappa),\mathrm{b}}_T(0, \mathbf{0}; T, \mathbf{y}) &= \frac{1}{\hat{C}_\nu} \left(\frac{2}{T} \right)^{2N(N+\nu)} h^{((4\nu-2\kappa+3)/4)}(\mathbf{y}^{\mathrm{odd}})^4 e^{-|\mathbf{y}^{\mathrm{odd}}|^2/T} \mathbf{1}(\mathbf{y} \in \mathbb{B}^{\mathrm{C}}_{2N}),
\end{aligned} \tag{50}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{W}^{\mathrm{C}}_{2N}, 0 \le s < t < T$, where $\hat{C}_\nu = 2^{N(2N+2\nu-1)} \prod_{i=1}^N \{\Gamma(2i)\Gamma(2(i+\nu))\}$ and $\mathbb{B}^{\mathrm{C}}_{2N} = \{\mathbf{x} = (x_1, x_2, \cdots, x_{2N}) : \mathbf{x}^{\mathrm{odd}} \in \mathbb{W}^{\mathrm{C}}_N, x_{2i} = x_{2i-1}, 1 \le i \le N\}$. We should notice that (50) includes the following

special cases.

$$g_T^{(\nu,0),\mathrm{b}}(0,\mathbf{0};T,\mathbf{y}) = q_\nu^{\mathrm{chGSE}}\left(\mathbf{y}^{\mathrm{odd}}; \frac{T}{2}\right) \mathbf{1}(\mathbf{y} \in \mathbb{B}_{2N}^{\mathrm{C}}) \quad \text{for} \quad \nu \in \mathbb{N},$$
$$g_T^{(-1/2,0),\mathrm{b}}(0,\mathbf{0};T,\mathbf{y}) = q^{\mathrm{DIII}}\left(\mathbf{y}^{\mathrm{odd}}; \frac{T}{2}\right) \mathbf{1}(\mathbf{y} \in \mathbb{B}_{2N}^{\mathrm{C}}).$$

Here

$$q_\nu^{\mathrm{chGSE}}(\boldsymbol{\kappa};t) = \frac{t^{-2N(N+\nu)}}{\hat{C}_\nu} \exp\left\{-\frac{|\boldsymbol{\kappa}|^2}{2t}\right\} \prod_{1\le i<j\le N} (\kappa_j^2-\kappa_i^2)^4 \prod_{k=1}^N \kappa_k^{4\nu+3}$$

is the probability density of the N distinct square roots $\boldsymbol{\kappa} = (\kappa_1,\kappa_2,\cdots,\kappa_N)$ with (11) of the eigenvalues of $M^\dagger M$ conditioned that M is a $2N\times 2N$ random matrices in the chiral Gaussian symplectic ensemble (chGSE) with variance t [54, 53, 27, 51], and

$$q^{\mathrm{DIII}}(\boldsymbol{\omega};t) = \frac{t^{-d[\mathrm{D}'']/2}}{C[\mathrm{D}'']} \exp\left\{-\frac{|\boldsymbol{\omega}|^2}{2t}\right\} \prod_{1\le i<j\le N} (\omega_j^2-\omega_i^2)^4 \prod_{k=1}^N \omega_k$$

with $d[\mathrm{D}''] = 2N(2N-1), C[\mathrm{D}''] = \hat{C}_{-1/2} = 2^{2N(N-1)} \prod_{i=1}^N \Gamma(2i)\Gamma(2i-1)$ is the probability density of the nonnegative and distinct eigenvalues $\boldsymbol{\omega} = (\omega_1,\omega_2,\cdots,\omega_N)$ with (18) of $4N\times 4N$ matrices in the ensemble in the symmetry class DIII studied by Altland and Zirnbauer [56, 1, 2]. (Strictly speaking, it is the DIII-even case. The DIII-odd case is obtained by setting $\nu = 1/2, \kappa = 0$ in (50).) The above implies that $\mathbf{X}^{(\nu,0),\mathrm{b}}(t)$ with $\nu \in \mathbb{N}$ and $\mathbf{X}^{(-1/2,0),\mathrm{b}}(t)$, both starting from $\mathbf{0}$, exhibit the transitions from the eigenvalue statistics of chGUE to chGSE and from the class D to the class DIII, respectively, as time t goes on from 0 to T.

A lengthy but explicit expression for the $4N\times 4N$ hermitian matrix-valued process corresponding to $\mathbf{X}^{(-1/2,0),\mathrm{b}}(t)$ is given as

$$\begin{aligned}
\Xi_T^{\mathrm{D,b}}(t) \quad = \quad & \sum_{\rho=0}^{2} \Big\{ \sqrt{-1} a_T^{0\rho}(t:O) \otimes (\sigma_0\otimes\sigma_\rho) + \sqrt{-1} a^{1\rho}(t) \otimes (\sigma_1\otimes\sigma_\rho) \\
& \quad + s_T^{2\rho}(t:O) \otimes (\sigma_2\otimes\sigma_\rho) + \sqrt{-1} a^{3\rho}(t) \otimes (\sigma_3\otimes\sigma_\rho) \Big\} \\
& + \Big\{ s^{03}(t) \otimes (\sigma_0\otimes\sigma_3) + s^{13}(t) \otimes (\sigma_1\otimes\sigma_3) \\
& \quad + \sqrt{-1} a_T^{23}(t:O) \otimes (\sigma_2\otimes\sigma_3) + s_T^{33}(t:O) \otimes (\sigma_3\otimes\sigma_3) \Big\},
\end{aligned}$$

$t \in [0,T]$, where $s^{\mu\rho}(t), s_T^{\mu\rho}(t:O) \in \mathcal{S}(N)$ and $a^{\mu\rho}(t), a_T^{\mu\rho}(t:O) \in \mathcal{A}(N)$, $t \in [0,T]$, are defined similarly to (17) and (41). Identification of its eigenvalue process with $\mathbf{X}^{(-1/2,0),\mathrm{b}}(t)$ gives the following version of Harish-Chandra integral,

$$\begin{aligned}
& \int_{U_1(4N)} dU \, \exp\left\{-\frac{1}{4\sigma^2} \mathrm{Tr}(\Lambda_{\mathbf{x}} - U^\dagger \Lambda_{\mathbf{y}} U)^2\right\} \\
= \quad & \frac{C_{2N}[\mathrm{D}]\sigma^{2N(4N+1)}}{h^{\mathrm{D}}(\mathbf{x}) h^{(1/4)}(\mathbf{y})^4} \det_{1\le i\le 2N, 1\le j\le N} \left[G^{\mathrm{D}}(\sigma^2, y_j|x_i) \quad \frac{x_i}{\sigma^2} G^{\mathrm{C}}(\sigma^2, y_j|x_i) \right]
\end{aligned}$$

for any $\sigma \in \mathbb{R}$, $\mathbf{x} = (x_1,x_2,\cdots,x_{2N}) \in \mathbb{W}_{2N}^{\mathrm{C}}, \mathbf{y} = (y_1,y_2,\cdots,y_N) \in \mathbb{W}_N^{\mathrm{C}}$, where $\Lambda_{\mathbf{x}} = \mathrm{diag}\{x_1,x_2,\cdots,x_{2N}\} \otimes \sigma_3$, $\Lambda_{\mathbf{y}} = \mathrm{diag}\{y_1,y_2,\cdots,y_N\} \otimes (\sigma_3\otimes\sigma_0)$, and $C_{2N}[\mathrm{D}] = (\pi/2)^N \prod_{i=1}^{2N} \Gamma(2i-1)$.

VI CONCLUDING REMARKS

In the present paper we showed that the eigenvalue processes of GUE, chGUE, the class C, and the class D are realized by the temporally homogeneous noncolliding diffusion processes and then the temporally inhomogeneous noncolliding diffusion processes were introduced, which exhibit the transitions in distribution from the eigenvalue statistics of GUE to GOE, GUE to GSE, chGUE to chGOE, chGUE to chGSE, the class C to the class CI, and the class D to the class DIII. They are obtained as the special cases of the noncolliding systems of the Brownian motions and those of Yor's generalized meanders. These inhomogeneous processes are identified with the eigenvalue processes of the inhomogeneous matrix-valued processes, some of which are regarded as the stochastic versions of two-matrix models studied by Pandey and Mehta [47, 41] as demonstrated in [31, 35]. We would like to put emphasis on the fact that in order to prove the identification we have not used any results by Pandey and Mehta, but used the generalized versions of Imhof relations ((4) and Proposition 8). Therefore we can give the proof for the Harish-Chandra (Itzykson-Zuber)-type integration formulae as corollaries. The present study suggests several open problems. Here we list up some of them.

(i) It does not seem to be possible to realize the eigenvalue processes of the random matrix ensembles different from GUE, chGUE, the class C and the class D by any temporally homogeneous noncolliding systems of diffusion particles. Is it possible to realize them as the temporally homogeneous diffusion processes with some conditions additional to the simple noncolliding condition ?

(ii) Norris, Rogers and Williams [46] studied other matrix-valued process called Dynkin's Brownian motion $\widetilde{\Xi}(t) = G(t)^T G(t)$ with $\partial G(t) = (\partial B(t))G(t)$, where ∂ denotes the Stratonovich differential; $x\partial y = xdy + dxdy/2$ for continuous semimartingales x, y. They showed that the eigenvalues of $\widetilde{\Xi}(t)$ are also noncolliding systems and derived the stochastic differential equations similar to (1) for the logarithms of the eigenvalues. As mentioned by Bru (see Remark 2 in [7]), $G(t)$ is a matrix-version of *multiplicative Brownian motion* in a sense, while $B(t)$ is the ordinary additional Brownian motion. Can we discuss (the logarithms of) the eigenvalue processes using the random matrix theory and noncolliding diffusion processes as well ?

(iii) In the non-hermitian random matrix ensembles, eigenvalues are distributed on the complex plane [20, 12]. Is it meaningful to consider the stochastic version of non-hermitian random matrix theory in the sense of Dyson [10] ?

For the temporally inhomogeneous noncolliding Brownian motions $\mathbf{X}(t), t \in [0, T]$ with $\mathbf{X}(0) = \mathbf{0}$, the determinantal expressions for the multi-time correlation functions were determined by Nagao and the present authors using the self-dual quaternion matrices [44, 16, 42] and the scaling limits of the infinite particles $N \to \infty$ and the infinite time-interval $T \to \infty$ were investigated [45, 30]. Recently Nagao reported the similar calculation on the process, which corresponds to the process $\mathbf{X}^{(1/2,1)}$ in the present paper [43]. Calculation of the multi-time correlations for the general process $\mathbf{X}^{(\nu,\kappa)}(t)$ is now in progress and the study of the infinite particle systems will be reported elsewhere [34].

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APPENDICES

A SCHUR FUNCTION EXPANSIONS OF DETERMINANTS

Any sequence $\mu = (\mu_1, \mu_2, \cdots, \mu_N, \cdots)$ of nonnegative integers in decreasing order $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq \cdots$ is called a partition. The non-zero μ_i in μ are called the parts of μ and the number of parts is the length of μ denoted by $\ell(\mu)$. For each partition μ with $\ell(\mu) \leq N$, the Schur function defined by $s_\mu(\mathbf{x}) = \det_{1\leq i,j\leq N}(x_i^{\mu_j+N-j})/\det_{1\leq i,j\leq N}(x_i^{N-j})$ gives a symmetric polynomial of order $|\mu| = \sum_{i=1}^N \mu_i$ in N variables $x_1, x_2, \cdots, x_N \in \mathbb{C}$. Note that the denominator is the Vandermonde determinant and $\det_{1\leq i,j\leq N}(x_i^{N-j}) = (-1)^{N(N-1)/2} h^{\mathrm{A}}(\mathbf{x})$ [39, 17, 52]. We can prove the following expansion formulae of the determinants with the bases of the Schur functions [3, 4, 36].

$$
\begin{aligned}
\frac{\det\limits_{1\leq i,j\leq N}\left[e^{x_i y_j}\right]}{h^{\mathrm{A}}(\mathbf{x})h^{\mathrm{A}}(\mathbf{y})} &= \sum_{\mu:\ell(\mu)\leq N} a_\mu s_\mu(\mathbf{x}) s_\mu(\mathbf{y}), \\
\frac{\det\limits_{1\leq i,j\leq N}\left[I_\nu(2\sqrt{x_i y_j})\right]}{\left\{\prod\limits_{i=1}^N x_i^{\nu/2} y_i^{\nu/2}\right\} h^{\mathrm{A}}(\mathbf{x})h^{\mathrm{A}}(\mathbf{y})} &= \sum_{\mu:\ell(\mu)\leq N} b_\mu^{(\nu)} s_\mu(\mathbf{x}) s_\mu(\mathbf{y}),
\end{aligned}
$$

where $a_\mu = 1/\prod_{i=1}^N \Gamma(\mu_i + N - i + 1)$ and $b_\mu^{(\nu)} = 1/\{\prod_{i=1}^N \Gamma(\mu_i + N - i + 1)\Gamma(\nu + \mu_i + N - i + 1)\}$. Since $s_\mu(\mathbf{0}) = \mathbf{1}(\mu = \mathbf{0})$ with $\mathbf{0} = (0, 0, \cdots, 0) \in \mathbb{N}^N$, from the above formulae, we have the following asymptotics of the determinants. As $|\mathbf{x}| \to 0$,

$$
\det_{1\leq i,j\leq N}\left[e^{x_i y_j}\right] = \frac{h^{\mathrm{A}}(\mathbf{x})h^{\mathrm{A}}(\mathbf{y})}{\prod_{i=1}^N \Gamma(i)} \times (1 + \mathcal{O}(|\mathbf{x}|)), \tag{A.1}
$$

$$
\det_{1\leq i,j\leq N}\left[I_\nu(2\sqrt{x_i y_j})\right] = \left\{\prod_{i=1}^N x_i^{\nu/2} y_i^{\nu/2}\right\} \frac{h^{\mathrm{A}}(\mathbf{x})h^{\mathrm{A}}(\mathbf{y})}{\prod_{j=1}^N \Gamma(j)\Gamma(\nu+j)} \times (1 + \mathcal{O}(|\mathbf{x}|)). \tag{A.2}
$$

B PROOF OF COROLLARY 12

By (38) of Lemma 5 (ii),

$$
\begin{aligned}
g_T^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}) &= \frac{T^{N(N+\kappa-1)/2} t^{-N(N+\nu)}}{C_{\nu,\kappa}} \exp\left\{-\frac{|\mathbf{y}|^2}{2t}\right\} h^{(2\nu+1)}(\mathbf{y}) \\
&\quad \times \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{z} \det_{1\leq i,j\leq N}\left[\frac{z_j^{\nu+1}}{y_i^\nu}\frac{1}{T-t}\exp\left\{-\frac{y_i^2+z_j^2}{2(T-t)}\right\} I_\nu\left(\frac{y_i z_j}{T-t}\right)\right] \times \prod_{k=1}^N z_k^{-\kappa} \\
&= \frac{T^{N(N+\kappa-1)/2} t^{-N(N+\nu)}}{(T-t)^N C_{\nu,\kappa}} \left(\frac{T}{t}\right)^{N(\nu+1-\kappa)} h^{(\nu+1)}(\mathbf{y}) \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{z} \exp\left\{-\frac{T}{2t^2}\left(\frac{t}{T}\right)^2 |\mathbf{z}|^2\right\} \\
&\quad \times \det_{1\leq i,j\leq N}\left[\exp\left\{-\frac{T}{2t(T-t)}\left(y_i^2 + \frac{t^2}{T^2}z_j^2\right)\right\} I_\nu\left(\frac{T}{t(T-t)}y_i \times \frac{t}{T}z_j\right)\right] \prod_{\ell=1}^N \left(\frac{t}{T}z_\ell\right)^{\nu+1-\kappa}.
\end{aligned}
$$

Setting $(t/T)z_i = a_i, 1 \leq i \leq N$, $t(1-t/T) = \sigma^2$ and $T/t^2 = \alpha$, we have

$$
\begin{aligned}
g_T^{(\nu,\kappa)}(0,\mathbf{0};t,\mathbf{y}) &= \frac{\sigma^{-2N}\alpha^{N(N+2\nu-\kappa+1)/2}}{C_{\nu,\kappa}} h^{(\nu+1)}(\mathbf{y}) \\
&\quad \times \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{a}\, e^{-\alpha|\mathbf{a}|^2/2} \det_{1\leq i,j\leq N}\left[e^{-(y_i^2+a_j^2)/2\sigma^2} I_\nu\left(\frac{y_i a_j}{\sigma^2}\right)\right] \prod_{\ell=1}^N a_\ell^{\nu+1-\kappa}.
\end{aligned} \tag{B.1}
$$

Proof of (i). We write the transition probability density of the process $M_T(t)$ by $Q_T(s, m_1; t, m_2)$, $0 \le s < t \le T$, for $m_1, m_2 \in \mathcal{M}(N+\nu, N; \mathbb{C})$. Then by Theorem 9 (i) and the fact (12),

$$g_T^{(\nu,\nu+1)}(0, \mathbf{0}; t, \mathbf{y}) = \frac{(2\pi)^{N(N+\nu)}}{C_\nu} h^{((2\nu+1)/2)}(\mathbf{y})^2 \int_{\mathrm{U}(N+\nu)\times\mathrm{U}(N)} d\mu(U,V)\, Q_T(0, O; t, U^\dagger K_{\mathbf{y}} V). \tag{B.2}$$

We introduce the $\mathcal{M}(N+\nu, N; \mathbb{C})$-valued process $M^{(1)}(t) = (m_{ij}^{(1)}(t))_{1\le i\le N+\nu, 1\le j\le N}$ and the $\mathcal{M}(N+\nu, N; \mathbb{R})$-valued process $M^{(2)}(t) = (m_{ij}^{(2)}(t))_{1\le i\le N+\nu, 1\le j\le N}$, whose elements are defined by

$$m_{ij}^{(1)}(t) = B_{ij}^0(t) - \frac{t}{T} B_{ij}^0(T) + \sqrt{-1}(\widetilde{\beta}_T^0)_{ij}(t) \quad \text{and} \quad m_{ij}^{(2)}(t) = \frac{t}{T} B_{ij}^0(T).$$

Then $M_T(t) = M^{(1)}(t) + M^{(2)}(t)$. Note that $\{B_{ij}^0(t) - (t/T)B_{ij}^0(T)\}$ are Brownian bridges of duration T starting at 0 and ending at 0, which are independent of $(t/T)B_{ij}^0(T)$. Hence $M^{(1)}(t)$ is in the chiral GUE distribution and $M^{(2)}(t)$ in the chiral GOE distribution, where $M^{(1)}(t)$ and $M^{(2)}(t)$ are independent from each other. Since $E[m_{ii}^{(1)}(t)^2] = \sigma^2$ and $E[m_{ii}^{(2)}(t)^2] = 1/\alpha$, $Q_T(0, O; t, M)$ for $M \in \mathcal{M}(N+\nu, N; \mathbb{C})$ can be written as

$$\begin{aligned} Q_T(0, O; t, M) &= \int_{\mathcal{M}(N+\nu, N; \mathbb{R})} \mathcal{V}(dB) \mu_\nu^{\mathrm{chGOE}}(B; 1/\alpha) \mu_\nu^{\mathrm{chGUE}}(M - B; \sigma^2) \\ &= \frac{\alpha^{N(N+\nu)/2} \sigma^{-N(N+\nu)}}{C_{\nu,\nu+1}(2\pi)^{N(N+\nu)}} \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{a}\, h^{(\nu)}(\mathbf{a}) e^{-\alpha|\mathbf{a}|^2/2 - \mathrm{Tr}(M - K_{\mathbf{a}})^\dagger (M - K_{\mathbf{a}})/2\sigma^2}, \end{aligned} \tag{B.3}$$

where we have used the fact (16) and the formulae (10), (15). Combining (B.1) with $\kappa = \nu + 1$, (B.2) and (B.3), we have

$$\begin{aligned} &\frac{C_\nu \sigma^{N(N+\nu-2)}}{h^{(\nu)}(\mathbf{y})} \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{a}\, e^{-\alpha|\mathbf{a}|^2/2} \det_{1\le i,j\le N} \left[\exp\left\{ -\frac{y_i^2 + a_j^2}{2\sigma^2} \right\} I_\nu\left(\frac{y_i a_j}{\sigma^2}\right) \right] \\ =& \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{a}\, h^{(\nu)}(\mathbf{a}) e^{-\alpha|\mathbf{a}|^2/2} \int_{\mathrm{U}(N+\nu)\times\mathrm{U}(N)} d\mu(U,V)\, e^{-\mathrm{Tr}(U^\dagger K_{\mathbf{y}} V - K_{\mathbf{a}})^\dagger (U^\dagger K_{\mathbf{y}} V - K_{\mathbf{a}})/2\sigma^2}. \end{aligned}$$

Since, for each $\sigma \in \mathbb{R}$, this equality holds for any $\alpha > 0$, we have the formula (i).
Proof of (ii). By setting $(\nu, \kappa) = (1/2, 1)$ and $(\nu, \kappa) = (-1/2, 0)$ in (B.1) we have the expressions for $\mathbf{x}, \mathbf{y} \in \mathbb{W}_N^{\mathrm{C}}$,

$$\begin{aligned} g_T^{(1/2,1)}(0, \mathbf{0}; t, \mathbf{x}) &= \frac{\alpha^{N(N+1)/2}}{C[\mathrm{C}']} h^{\mathrm{C}}(\mathbf{x}) \int_{\mathbb{W}_N^{\mathrm{C}}} d\mathbf{a}\, e^{-\alpha|\mathbf{a}|^2/2} \det_{1\le i,j\le N} \left[G^{\mathrm{C}}(\sigma^2, a_j | x_i) \right], \\ g_T^{(-1/2,0)}(0, \mathbf{0}; t, \mathbf{y}) &= \frac{\alpha^{N^2/2}}{C[\mathrm{D}']} h^{\mathrm{D}}(\mathbf{y}) \int_{\mathbb{W}_N^{\mathrm{D}}} d\mathbf{a}\, e^{-\alpha|\mathbf{a}|^2/2} \det_{1\le i,j\le N} \left[G^{\mathrm{D}}(\sigma^2, a_j | y_i) \right]. \end{aligned}$$

Following the same argument with the proof of (i) and using the equalities (19) and (23), the formulae (ii) are proved. ∎

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